B. Ticknor

A

TREATISE

ON THE

CONSTRUCTION, PROPERTIES, AND ANALOGIES

OF THE

THREE CONIC SECTIONS.

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BY THE

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The present edition of Bridge's Conic Sections is reprinted, with a few alterations, from the second London Edition. Such changes only have been made, as seemed necessary to adapt the work to the purpose for which it is intended, namely, that of furnishing a text-book for recitation in Colleges. To this end, the propositions have been enunciated without the use of letters and without reference to particular diagrams. As it is true, however, that a proposition is more readily comprehended, when it is asserted of the lines and angles of some particular figure, immediately before the eye, than when expressed in general terms; it has been thought proper to introduce the general enunciations at the close of the demonstrations to which they belong, and to leave the author's statements at the commencement of those demonstrations unaltered. In this respect the convenience of students has heen consulted, rather than the usual practice of writers.

The demonstrations, as they stand in the original work of Bridge, are in general, so much distinguished for conciseness and simplicity, and leave so little to be supplied by the student, (a circumstance of great importance in a book designed for large classes,) that it has been thought best to vary from the author only in a very few instances. Alterations have, however, been made, when they seemed likely to be attended by any material advantage.

The number of the propositions has been somewhat increased, not for the purpose of completing the enumeration of particular properties of the Conic Sections, which, in a work like this, considering the fertility of the subject, would be equally impracticable and useless; but in order to exhibit to the student how far many truths may be generalized, which he is apt to consider as limited by particular circumstances. A few propositions, not of this kind, have been added, as

being among the more curious of those which Bridge has omitted to notice.

For the convenience of students, some references, particularly towards the close of the book, have been made to the mathematical treatises of President Day.

The original numbering of the Properties and of the Articles has been suffered to stand; and whenever any thing has been inserted in the body of the work, the number of the preceding article has been repeated with a letter annexed. The additional Properties are distinguished by the capitals A, B, C, &c. A few notes contain whatever else is peculiar to this Edition.

F. A. P. BARNARD.

Yale College, June 20, 1831.

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CONIC SECTIONS.

CHAPTER I.

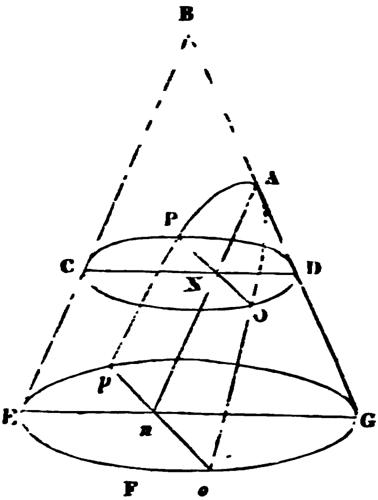
INTRODUCTION.

A CONE is a solid figure formed by the revolution of a right angled triangle about one of its sides. (Euc. Def. 11. 3. Sup.) From the manner in which this solid is generated, it is evident that if it be cut by a plane parallel to its base, the intersection of the plane with the solid, will be a *circle*, since this section will coincide with the revolution of a perpendicular to the fixed side of the triangle; and if it be cut by a plane passing through its vertex, the intersection will be a triangle, the sides of which will correspond to the hypothenuse of the generating triangle, in different positions, or at different periods of the revolution. If the plane by which the cone is cut be not parallel to the base, or do not pass through the vertex, then the line traced out upon its surface will be one of those curves more particularly distinguished by the name of CONIC SECTIONS, the properties of which are to be made the subject of the following Treatise.

I.

(1.) Let BEFGp be a cone, and let it be cut by a plane BEnG perpendicular to its base and passing through its vertex; then the section BEG will be a *triangle*. Next, let it be cut by a plane pA$o$$n$ at right angles to the plane BEnG, *and parallel to a plane touching the side* BE *of the cone;* then the curve line pPAOo, which is formed by the intersection of this latter plane with the surface of the cone, is called a *Parabola*.

For the purpose of investigating the nature of this curve, let CPDON be a plane parallel to the base of the cone; the intersection CPDO of this plane with the cone will be a *circle*. Since the plane BEnG divides the cone into two equal parts, CD (the common intersection of the planes BEnG, CPDON) will be a diameter of that circle; and for the same reason EG will be a diameter of the circle EpGoF. Let ANn be the common intersection of the planes BEnG, pAon, and PNO, pno, those of the plane pAon, with the planes CPDON, EpGoF respectively. Because the planes pAon,

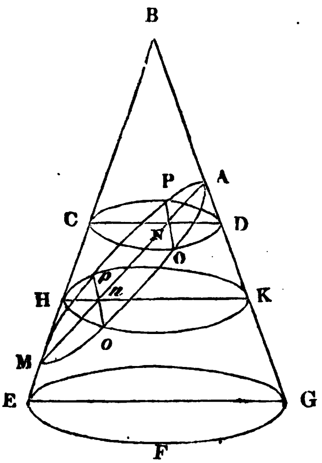


CPDON are perpendicular to the plane BEnG, PNO must be perpendicular to the plane BEnG, (Euc. 18. 2. Sup.) and consequently perpendicular to the two lines AN, ND drawn in that plane; (Euc. Def. 1. 2. Sup.) for the same reason pno is perpendicular to the two lines An, nG. Hence by the property of the circle CN $\times$ ND$=$PN^2, or ND$=\frac{PN^2}{CN}$; and E$n\times n$G$=pn^2$, or nG$=\frac{pn^2}{En}$.

Now since An is parallel to BE, and CD parallel to EG, the figure CNnE is a parallelogram; $\therefore$CN$=$En. By similar triangles AND, AnG; AN : An::ND : nG::$\frac{PN^2}{CN}:\frac{pn^2}{En}$::(since CN$=En$) PN^2 : pn^2.

(2.) Hence the nature of the curve APp is such, that if it begins to be generated from the given point A, and PN is drawn always at right angles to AN, AN will vary as PN^2. And the same may be said with respect to the relation of AN and NO on the other side of ANn.

(3.) Next, let the plane MPAoM be drawn, as before, perpendicular to the plane BEG, *but passing through the sides of the cone* BE, BG; then the curve MPAoM, formed by the intersection of this plane with the surface of the cone, is called an *Ellipse.*



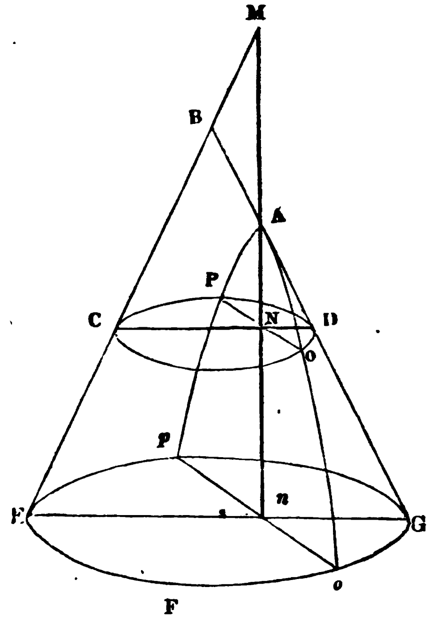
In this case, draw *two* planes, CPDON, HpKon, parallel to the base of the cone; then, for the same reason as before, PN will be perpendicular both to AN and ND, and pn will be perpendicular both to An and nK; $\therefore NC \times ND = PN^2$, and $nH \times nK = pn^2$.

By sim. triangles AND, AnK; MNC, MnH;

$$\begin{array}{llllllll} \text{we have} & AN & : & An & :: & ND & : & nK, \\ & NM & : & nM & :: & NC & : & nH; \\ \hline \therefore & AN \times NM & : & An \times nM & :: & NC \times ND & : & nH \times nK \\ & & & & :: & PN^2 & : & pn^2. \end{array}$$

(4.) The nature of the curve APM therefore is such, that if A and M are given points, and PN be always drawn at right angles to AM *between the points* A, M, $AN \times NM$ will vary as PN^2; and the same with respect to the relation between $AN \times NM$ and NO^2.

(5.) Lastly, let the plane $pAon$ be drawn, as before, perpendicular to the plane BEG, but *cutting the side* BG *in* A, and, when produced, *meeting a plane drawn touching the other side* EB *produced, in* M; then the curve $pPAOo$ formed by the intersection of the plane $pAon$ with the surface of the cone, is called an *Hyperbola.*



Let the plane CPDON be drawn parallel to the base; then, by similar triangles, AND, AnG; MNC, MnE; we have

$$\begin{array}{ccccccc} AN & : & An & :: & ND & : & nG, \\ NM & : & nM & :: & NC & : & nE; \end{array}$$

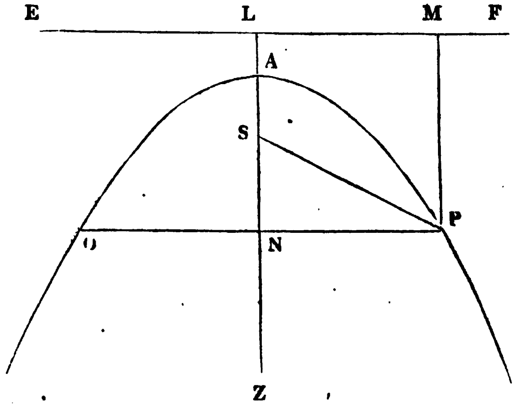
$$\therefore AN \times NM : An \times nM :: ND \times NC : nG \times nE :: PN^2 : pn^2$$

(6.) Hence the nature of the curve APp is such, that if A and M are given points, and PN be always drawn at right angles to AN, *the point* A *lying between* M *and* N, then $AN \times NM$ will vary as PN^2; and the same with respect to the relation of $AN \times NM$ and NO^2.

II.

Having thus explained the nature of the curves arising from the intersection of a plane with the surface of a cone, we now proceed to show how these curves may be constructed geometrically.

(7.) Let ELF be a line given in position, and LZ another line drawn at right angles to it in the point L. In LZ take any point



S, and bisect SL in A. Let a point P move from A, in such a manner that it may always be at equal distances from S and the line ELF (or, in other words, let the line SP revolve round S as a center, and intersect another line PM moving parallel to LZ, in such a manner that SP may be always equal to PM ;) then the point P will trace out a curve OAP, having two similar branches AP, AO, one on each side of the line AZ; which curve will be a *Parabola.*

To show that this curve will be a parabola, draw PNO at right angles to AZ; then LNPM will be a parallelogram, and LN=PM =SP; but LN=AN+AL=AN+AS (since AL=AS by construction,) $\therefore$SP=AN+AS.

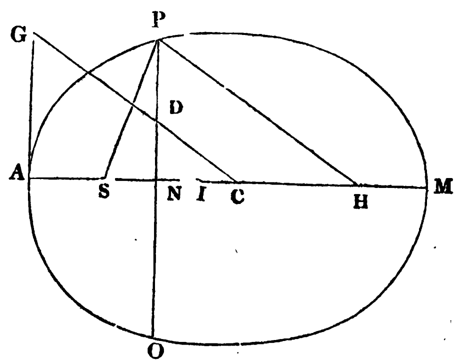
Let $AN=x$,
$PN=y$,
$SA=a$;
then $SP=AN+AS$
$=x+a$,
and $SN=AN-AS$
$=x-a$.

Now $PN^2=SP^2-SN^2$, (Euc. 47. 1.)
or $y^2=(x+a)^2-(x-a)^2$
$=x^2+2ax+a^2-x^2+2ax-a^2$
$=4ax$.

Since $4a$ is a constant quantity, x varies as y^2, or AN $\propto$ PN^2; the relation between AN and PN is therefore the same as in Art. 2.; hence the curve AP is a Parabola.*

(8.) Next, take any line SH, and produce it both ways towards A and M. Let a point P begin to move from A, in such a manner that the *sum* of its distances from S and H may be always the same, (or, in other words, let two lines, SP, PH, intersecting each other in P, revolve round the fixed points S and H, in such a manner that SP+PH may be a constant quantity;) then the curve APMO traced out by the point P will be an Ellipse.

To prove this, it may be observed that when P is at A, then HA+AS or HS+2AS, is equal to that constant quantity; and when P is at M, SM+MH or HS+2HM, is equal to the same



quantity. Hence HS+2AS=HS+2HM, from which it appears that 2AS=2HM, or AS=HM. Now SP+PH=HA+AS= HA+HM=AM; bisect therefore SH in C, and make CM equal to

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* Geometrically demonstrated thus,
Since SP=AN+AS, SP^2=(AN+AS)^2= (Euc. 8. 2.) SN^2+ 4AS.AN. But (Euc. 47. 1.) SP^2=SN^2+PN^2; ∴SN^2+PN^2= SN^2+4AS.AN, or PN^2=4AS.AN.

CA, then M will be the point where the curve cuts the line SH produced; and AM will be the constant quantity to which SP+PH is equal.

Let	Then	
AC or $CM=a$,	$AN=AC-CN=a-x$,	and SP+PH=
SC or $CH=b$,	$NM=CM+CN=a+x$,	$AM=2AC=2a$; if
$CN=x$,	$SN=SC-CN=b-x$,	$\therefore SP=a-z$, HP will
and $PN=y$;	$NH=CH+CN=b+x$,	be equal to $a+z$.

Draw PNO at right angles to AM, then (Euclid, 47. 1.) we have,

$$HP^2=PN^2+NH^2, \text{ or } (a+z)^2=y^2+(b+x)^2, \quad (A)$$

and $$SP^2=PN^2+NS^2, \text{ or } (a-z)^2=y^2+(b-x)^2. \quad (B)$$

Subtract (B) from (A), then $4az=4bx$, or $z=\frac{bx}{a}$; substitute this value for z in equation (A), and it becomes

$$\left(a+\frac{bx}{a}\right)^2=y^2+(b+x)^2,$$

which reduced is

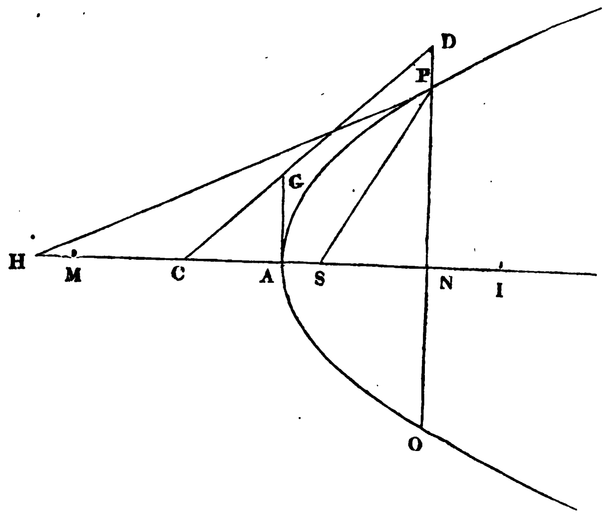
$$a^4+2a^2bx+b^2x^2=a^2y^2+a^2b^2+2a^2bx+a^2x^2,$$

$$\text{or } a^4-a^2b^2-a^2x^2+b^2x^2=a^2y^2,$$

$$\text{i. e. } (a^2-b^2)\times(a^2-x^2)=a^2y^2.$$

But since a and b are constant quantities, a^2-x^2 varies as y^2; now $a^2-x^2=(a-x)\times(a+x)$; $\therefore (a-x)\times(a+x)\propto y^2$, or $AN\times NM \propto PN^2$; hence, as N lies between A and M, the relation between $AN\times NM$ and PN^2 is such, that the curve APM is an Ellipse.

(9.) Lastly, Take any line SH, and let the two lines SP, HP, intersecting each other in P revolve round the fixed points, S, H, in such a manner that the *difference* of the lines HP and SP (viz. HP−SP) may be a constant quantity; then the curve traced out by the point P will be an Hyperbola.



In this case, let A be the point where the curve cuts SH; bisect SH in C, and take CM=CA. Since CH=CS, and CM=CA, HM will be equal to AS. Now when P comes to A, HA−AS= a constant quantity; but HA−AS=HA−HM=AM; ∴AM is that constant quantity. Hence AM=HP−SP.

$$\begin{array}{lll} \text{Let} & \text{Then} & \\ \left.\begin{array}{r} AC \text{ or } CM=a, \\ SC \text{ or } CH=b, \\ CN=x, \\ \text{and } PN=y; \end{array}\right\} & \begin{array}{l} AN=CN-CA=x-a, \\ NM=CN+CM=x+a, \\ NS=CN-CS=x-b, \\ NH=CN+CH=x+b, \end{array}\Bigg\} & \begin{array}{l} \text{and } HP-SP= \\ AM=2AC=2a; \\ \text{if } \therefore HP=z+a, \\ SP \text{ will be equal} \\ \text{to } z-a. \end{array} \end{array}$$

Draw PNO at right angles to AN, then we have

$$HP^2=PN^2+NH^2, \quad \text{or} \quad (z+a)^2=y^2+(x+b)^2, \quad \text{(A)}$$
$$SP^2=PN^2+NS^2, \quad \text{or} \quad (z-a)^2=y^2+(x-b)^2. \quad \text{(B)}$$

Subtract (B) from (A), then $4az=4bx$, and $z=\frac{bx}{a}$; substitute this value for z in equation (A), and there results $\left(\frac{bx}{a}+a\right)^2=y^2+(x+b)^2$, which reduced is

$$b^2x^2+2a^2bx+a^4=a^2y^2+a^2x^2+2a^2bx+a^2b^2,$$
$$\text{or } b^2x^2-a^2x^2-a^2b^2+a^4=a^2y^2,$$
$$\text{i. e. } (b^2-a^2)\times(x^2-a^2)=a^2y^2.$$

Hence $x^2-a^2 \propto y^2$, or $(x-a)\times(x+a) \propto y^2$; i. e. $AN \times NM \propto PN^2$; and since A lies between N and M, the relation between $AN\times NM$, and PN^2, is the same with that in the Hyperbola.*

Having thus established the identity of the curves generated by these two different methods, we now proceed to demonstrate their properties, beginning with the parabola.

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* The same may be proved geometrically, as follows. The demonstration is applicable either to the Ellipse or Hyperbola.

Take $AI=SP$. Then $IM=HP$. $\therefore HP=CI+CA$, and $SP=CI\sim CA$.

Now (Euc. 47.1.) $(CI+CA)^2(=HP^2)=PN^2+(CN+CS)^2(=HN^2)$; and, $(CI\sim CA)^2(=SP^2)=PN^2+(CN\sim CS)^2(=SN^2)$. That is, $CI^2+2CA.CI+CA^2=PN^2+CN^2+2CN.CS+CS^2$, and $CI^2-2CA.CI+CA^2=PN^2+CN^2-2CN.CS+CS^2$. Subtract, and $4CA.CI=4CN.CS$ or $CA.CI=CN.CS$, $\therefore CA:CN::CS:CI$, and $CA^2:CN^2::CS^2:CI^2$.

From A, draw AG at right angles to AC; make AG a mean proportional between AS and SM, and join CG, meeting PN in D. Then $AG^2=AS.SM=CS^2\sim CA^2$, (Euc. 5. 2. cor.) and $CS^2=CA^2\pm AG^2$.*

By sim. tri. $CA^2:CN^2::CA^2\pm AG^2(CS^2):CN^2\pm ND^2$,* but (as above) $CA^2:CN^2::CS^2:CI^2 \therefore CI^2=CN^2\pm ND^2$.

In the first equation as expanded above, therefore, let $CS^2\pm AG^2$† be substituted for CA^2, $CS.CN$ for $CA.CI$, and $CN^2\pm ND^2$ for CI^2, and we have $CS^2\pm AG^2+2CS.CN+CN^2\pm ND^2=PN^2+CS^2+2CS.CN+CN^2$, or $\pm AG^2\pm ND^2=PN^2$, that is $AG^2\sim ND^2=PN^2$. But (sim. tri.) $AC^2:AG^2::CN^2\sim CA^2(AN.NM):AG^2\sim ND^2(PN^2)$. But the ratio $AC^2:AG^2$ is constant. Hence $AN.NM \propto PN^2$, which (N being between A and M) is the property of the Ellipse, and (A being between N and M) is the property of the Hyperbola.

* The sign $-$ for the Ellipse, and $+$ for the Hyperbola.

† The sign $+$ for the Ellipse, and $-$ for the Hyperbola.

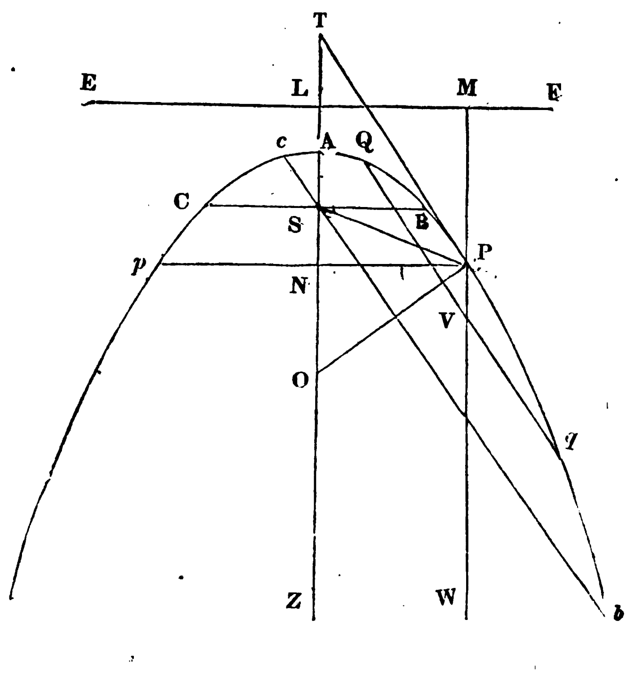
CHAPTER II.

ON THE PARABOLA.

III.

DEFINITIONS.

(10.) Let pAP be a parabola generated by the lines SP, PM, moving according to the law prescribed in Art. 7.; then the line ELF, which regulates the motion of the line PM, is called the *Directrix;* the point S, about which the line SP revolves, the *Focus;* the line AZ; which passes through the middle of the curve, the *Axis;* and the highest point A, the *Vertex* of the parabola.



(11.) Let fall the perpendicular PN upon the axis AZ, and through the focus S draw BC parallel to it, and meeting the curve in the points B and C. PN is then called the *Ordinate* to the axis, AN the *Abscissa;* and the line BC is called the *Principal Latus-rectum*, or the *Parameter to the Axis.*

(12.) Produce MP in the direction PW, or, in other words, draw PW parallel to the axis AZ; from any point Q of the parabola draw QVq parallel to a tangent at P; and through S draw *bc* parallel to QV. PW is called the *diameter* to the point P; QV the *ordinate*, PV the *abscissa*, and *bc* the *parameter*, to the diameter PW.

(13.) Let PT touch the curve in P, and meet the axis produced in T, draw PO at right angles to PT, and let it cut the axis in O. PT is called the *tangent*, TN the *subtangent*, PO the *normal*, and NO the *subnormal*, to the point P.

IV.

On the Properties of the Parabola.

Property 1.

(14.) The Latus-rectum BC is equal to 4AS.

Draw BD (Fig. in page 13.) parallel to LZ, then SB=BD=SL. But since SA=AL, SL is equal to 2AS; hence SB=2AS, and 2SB or BC=4AS.

This proposition may be thus enunciated.

The latus-rectum is equal to four times the distance from the focus to the vertex.

Property 2.

(15.) The tangent PT bisects the angle MPS.

Take P*p* so small a part of the curve, that it may be considered as coinciding with the tangent, and consequently as a right line. Join S*p*, and draw *pm* parallel to AZ; let fall *po*, *pn*, perpendiculars upon SP, PM.

The figure $Mnpm$ is a parallelogram, $\therefore nM=pm$; and since po is at right angles to SP, it may be considered as a small circular arc described with radius Sp, $\therefore So=Sp$. Also $SP=PM$, and $Sp=pm$.

$$\left.\begin{aligned}\text{Now } Po&=SP-So=SP-Sp,\\ \text{and } Pn&=PM-nM=SP-pm,\\ &=SP-Sp,\end{aligned}\right\}\therefore Po=Pn.$$

In the small right-angled triangles Ppo, Ppn, we have therefore Pp common, and $Po=Pn$, $\therefore$ (47. 1.) $po=pn$; having $\therefore$ their three sides equal, the angle pPo must be equal to the $\angle pPn$; hence, since pT may be considered as the continuation of the line Pp, PT bisects the angle MPS; which proposition may be thus expressed.

The tangent, at any point of the curve, bisects the angle formed at that point, by the perpendicular to the directrix, and the line drawn to the focus.*

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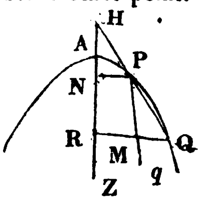
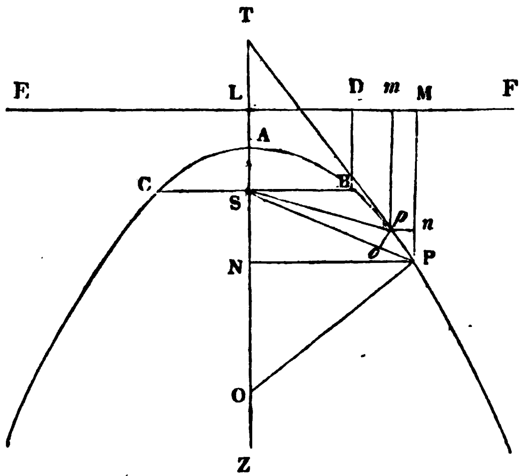
* The reasoning in the text, though perfectly conclusive, is of a kind not always entirely satisfactory to the student, who is unaccustomed to its use. The same proposition may be demonstrated without the use of indefinitely small arcs, in the following manner.

It is first necessary to establish this position:—If a straight line, not parallel to the axis of the parabola, cut the curve in one point, it will, on being produced, if necessary, cut it again.

Let HP, not parallel to AZ, the axis, cut the curve in P. It will, on being produced towards P, intersect the curve in some other point.



Since HP and AZ are not parallel, they will meet, if produced. Let them meet in H. Draw the ordinate PN, and take AR a third proportional to AN and AH. Draw the ordinate RQ. HP, produced, will meet the curve in Q.



(16.) Cor. Since the angle MPS continually increases as P moves towards A, and at A becomes equal to two right angles, the tangent at A must be perpendicular to the axis.

For, if not, let it take some other direction, as P*q*, cutting RQ in M.*

By Hyp. $AN : AH :: AH : AR$ (Euc. 12. 5.) $AN : AH :: AN+AH(NH) : AR+AH(RH)$ and $AN^2 : AH^2 :: NH^2 : RH^2 ::$ (sim. tri.) $PN^2 : MR^2$ (Euc. Def. 11. 5.) $AN^2 : AH^2 :: AN : AR ::$ (7.) $PN^2 : QR^2$, $\therefore PN^2 : MR^2 :: PN^2 : QR^2$, or $MR^2 = QR^2$, which is impossible, unless HP produced pass through Q. Therefore, &c.

Cor. Hence if HQ cuts the curve, $AN : AH :: AH : AR$.

The demonstration is not essentially different for any other arrangement of the points A, P and Q.

To prove that the tangent bisects the angle SPM, let the ordinate PN be drawn.

---

* M is taken between Q and R, because, if taken on the other side of Q, P*q* *must* cut the curve.

PROPERTY 3.

If PT meets the axis produced in T, then SP=ST, and TN=2AN.

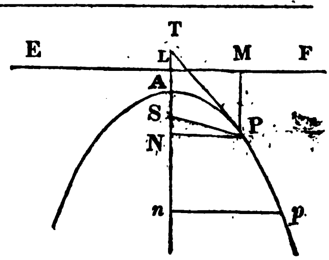
(17.) Since PM is parallel to TZ, the angle MPT=alternate angle STP; but (by Prop. 2.) $\angle$MPT=$\angle$SPT, $\therefore\angle$STP=$\angle$SPT, and consequently SP=ST. That is

If a tangent to any point of the curve cut the axis produced, the points of contact and intersection will be equally distant from the focus.

(18.) Now (7.) SP=AN+AS;
and ST=TA+AS.

Hence, since SP=ST, we have AN+AS=TA+AS, $\therefore$TA=AN, or TN=2AN.

Now if the tangent does not bisect SPM, some other line which *cuts* the curve, must do it. Let TP be that line, cutting the curve in P and again in *p*.* Draw the ordinate *pn*.



By Hyp. $\angle$SPT=$\angle$TPM=alternate$\angle$STP. $\therefore$(Euc. 5. 1.) ST=SP=PM(7.)=LN.

From ST=LN take AS=(7.)AL, and AT=AN. But by the corollary above, AN : AT : : AT : A*n*, or AN=A*n*, which is absurd, if TP cuts the curve, $\therefore$TP is the tangent.

Hence the tangent bisects $\angle$SPM.

It will be seen that while we are demonstrating Property 2d, we at the same time prove all that is laid down in Arts. 17 and 18. Indeed it would be better to demonstrate these latter propositions first, and infer Property 2d from them.

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* It is not essential to the demonstration, on which side of P, *p* is taken.

The subtangent is bisected by the vertex; or the subtangent is double the corresponding abscissa.

(18*a*.) Hence the tangent at C, the extremity of the latus-rectum meets the axis in L, the same point with the directrix. For (7.) SA=AL. Hence SL=2SA=CS, (14.) and the triangle CSL is isoceles.

PROPERTY 4.

(19.) The square of the ordinate (PN^2)= latus-rectum × abscissa (BC × AN.)

By Art. 7, $y^2=4ax$, or $PN^2=4AS \times AN$; but by Prop. 1, BC=4AS, $\therefore PN^2=BC \times AN$. Or, the square of any ordinate to the axis is equal to the rectangle of the corresponding abscissa and the latus-rectum.

(20.) COR. Hence $BC=\frac{PN^2}{AN}$; and $\frac{1}{2}BC=\frac{PN^2}{2AN}$.

PROPERTY 5.

(21.) The subnormal $NO=\frac{1}{2}BC$.

Since TPO is a right angled triangle, (Euc. 8. 6. cor.) NO : PN : : PN : TN, $\therefore NO=\frac{PN^2}{TN}$; but by Prop. 3, TN=2AN, $\therefore NO=\frac{PN^2}{2AN}$; hence (20.) $NO=\frac{1}{2}BC$. Or, the subnormal is equal to half the latus-rectum.

PROPERTY 6.

(22.) The square of the ordinate (QV^2)=4SP×PV. (See next figure.)

Produce VQ to H; draw EQ, GV parallel to PN, and QD parallel to AZ; then the figures PTHV, PNGV will be parallelograms, and TH=PV=NG; ∴ HN+NG=HN+TH, or HG=TN.

$$\left.\begin{array}{l} \text{Let } AN=x, \\ PV=NG=y, \\ QD=EG=z, \\ \quad AS=a, \\ \text{and} \therefore SP=AN+AS=x+a. \end{array}\right\} \text{Then} \left\{\begin{array}{l} HG=TN=2AN=2x \\ HE=HG-GE=2x-z \\ AG=AN+NG=x+y \\ AE=AG-EG=x+y-z. \end{array}\right.$$

Now (19.) $EQ^2=4AS\times AE=4a\times(x+y-z)$; and by similar triangles, HEQ, TNP, we have

$$HE^2 : EQ^2 :: TN^2 : PN^2,$$

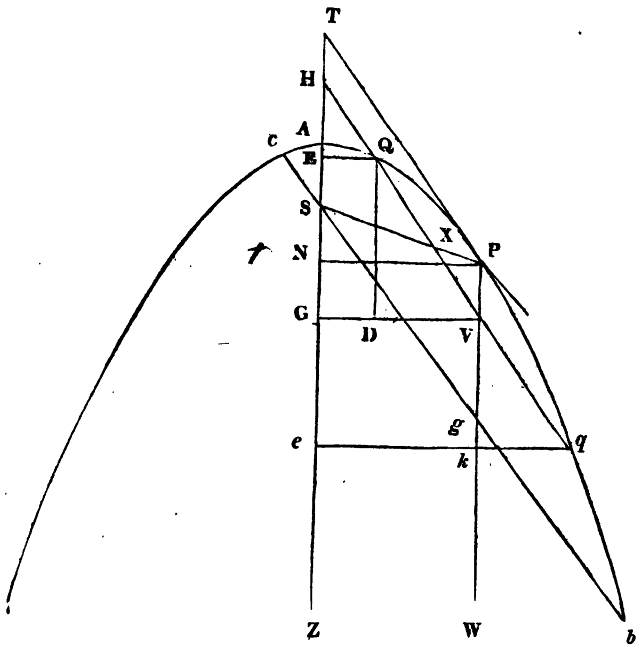
i. e. $(2x-z)^2 : 4a\times(x+y-z) :: 4x^2 : 4ax,$

$$\therefore (2x-z)^2=\frac{4a\times(x+y-z)\times 4x^2}{4ax}$$

$$=4x\times(x+y-z),$$

$$\text{or } 4x^2-4xz+z^2=4x^2+4xy-4xz.$$

$$\text{Hence } z^2=4xy=QD^2.$$



Again, by sim. Δs, HGV, QDV, we have

$$HG^2 : GV^2 \text{ or } PN^2 :: QD^2 : DV^2,$$

$$\text{i. e. } 4x^2 : 4ax :: 4xy : DV^2 = \frac{4ax \times 4xy}{4x^2} = 4ay.$$

But $QV^2 = QD^2 + DV^2$

$$= 4xy + 4ay = 4(x + a)y = 4SP \times PV.$$

That is, the square of an ordinate to any diameter, is equal to four times the rectangle of the corresponding abscissa, and the distance from the vertex of that diameter to the focus.*

---

* This proposition may be demonstrated geometrically as follows. In the first place $QV^2 \propto PV$.

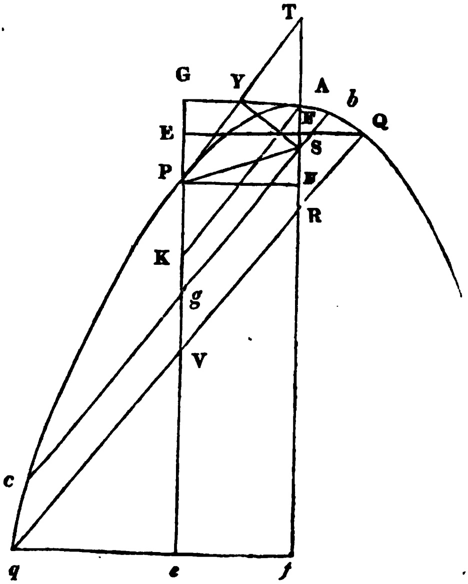
From A draw AK, an ordinate to PV, and AG, the vertical tangent. Putting L for the latus-rectum, we have (19) L.AN (or AT) $= PN^2 = EF^2$, and L.AF $= FQ^2$.

Again, by sim. tri. PTN, QRF

$$TN(=2AN) : FR :: PN(=EF) : FQ,$$

$$\text{or } AN : FR :: EF : 2FQ.$$

And the rect's $L.AN : L.FR :: EF^2 : 2EF.FQ$.



The same demonstration, with a very slight alteration, is applicable to the case when P and Q are on opposite sides of A.

PROPERTY 7.

(23.) If QV is produced to meet the curve in q, then qV=QV. (Fig. on p. 16.)

Draw qe at right angles to AZ, cutting PW in k; then He=HG+Ge, and Ae=AG+Ge; if therefore Ge or Vk=z, we have He=

But $L.AN=EF^2$ $\therefore$ $L.FR=2EF.FQ$.

To this add $L.AT=EF^2$

$L.AF=FQ^2$

$\therefore$ $L(TA+AF+FR)=L.TR=EF^2+2EF.FQ+FQ^2=EQ^2$.

But (sim. tri.)$AK^2 : QV^2 :: AG^2(L.AT) : EQ^2(L.TR)$,

or $AK^2 : QV^2 :: AT : TR :: PK : PV$.

But since AK^2 and PK are constant $QV^2 \propto PV$.

Next, $QV^2=4SP.PV$.

Upon the tangent PT, let fall the perpendicular SY, from the focus. Since STP is isosceles, PT is bisected by SY. AY also bisects PT, since AT=AN, (Euc. 2. 6.) Hence AY and SY intersect the tangent PT, in the same point.

By sim. tri. $PN^2 : PT^2(AK^2) :: SY^2(=AS.ST$ Euc. 8. 6. cor.) $: ST^2 :: AS : ST(SP)$.

Hence, because $AN=AT=PK$ and $4AN=4PK$,

$PN^2 : AK^2 :: 4AS.AN : 4SP.PK$.

But $PN^2=4AS.AN$. $\therefore$ $AK^2=4SP.PK$.

But $AK^2 : QV^2 :: PK : PV :: 4SP.PK : 4SP.PV$.

$\therefore$ $QV^2=4SP.PV$.

These demonstrations are equally applicable to qV. The first will require one change of sign, $(-ef.fq)$ but in all other respects it may remain the same, only substituting the small for the large letters. And as eq^2 is proved, in this way, equal to $L.TR=EQ^2$, by sim. tri. QV=qV, and Prop. 7 requires no additional demonstration.

$2x+z$, and $Ae=x+y+z$. In this case, the sign of z is changed throughout; reasoning therefore as in the last Property, we should have z^2 or $Vk^2=4xy$.

By sim. $\triangle$s, HGV, Vkq, $HG^2 : GV^2$ or $PN^2 :: Vk^2 : kq^2$,
i. e. $4x^2 : 4ax :: 4xy : kq^2$.

$$\text{Hence } kq^2=\frac{4ax\times 4xy}{4x^2}=4ay.$$

$$\begin{aligned}\text{but } Vq^2&=Vk^2+kq^2\\ &=4xy+4ay=4(x+a)y=4SP\times PV.\end{aligned}$$

Since QV^2 and Vq^2 are each equal to $4SP\times PV$, it follows that $Vq^2=QV^2$, and consequently $Vq=QV$. Or, Every diameter bisects all lines in the parabola, drawn parallel to the tangent at its vertex, and terminated both ways by the curve; or every diameter bisects its double ordinates.

Property 8.

(24.) The Parameter bc is equal to 4SP. (Fig. on p. 16.)

Let bc and PW intersect each other in g, then (23) $cg=gb$. $\therefore$ $cg=\frac{1}{2}bc$, and $cg^2=\frac{1}{4}bc^2$.

Since $PgST$ is a parallelogram, $Pg=ST=SP$. Now (22) $cg^2=4SP\times Pg=4SP\times SP=4SP^2$.

$$\begin{aligned}\text{Hence } \tfrac{1}{4}bc^2&=4SP^2,\\ \text{and } \tfrac{1}{2}bc&=2SP,\\ \text{or } bc&=4SP;\end{aligned}$$

that is, the parameter to any diameter is equal to four times the distance from the vertex of that diameter to the focus.

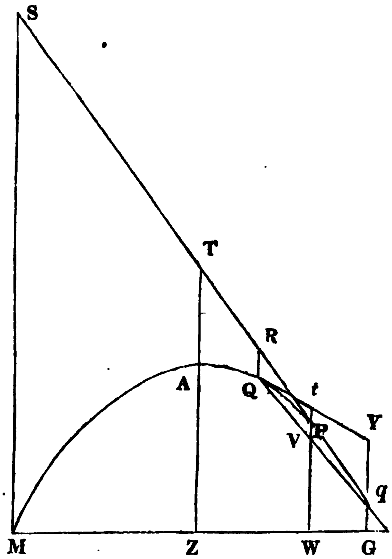
(25.) Cor. Hence $QV^2=(4SP\times PV=)bc\times PV$; and since bc is constant with respect to the same diameter, $PV\propto QV^2$. That the square of the ordinate is equal to the parameter $\times$ abscissa, is therefore a *general property* of the Parabola.

Property 9.

(26.) Draw QR parallel to PV, and meeting PT in R; then $QR\propto PR^2$.

In this case (since QV is parallel to RP) PVQR is a parallelogram; $\therefore$ PR = QV, and QR = PV; but (25) PV $\propto QV^2$, $\therefore QR \propto PR^2$. Or, if diameters be produced to meet any tangent to the Parabola, without the curve, the parts of those diameters between the curve and the tangent will be as the squares of the intercepted parts of the tangent.

(27.) Cor. From this it follows, that if PS, PW, be two lines meeting in a given angle, and a point Q begins to move from P in such a manner that its distance RQ from the line PS (measured in a direction parallel to PW) shall vary as PR^2, or in other words, that RQ, TA, SM, &c. shall be to each other as PR^2, PT^2, PS^2, &c. then the curve PQAM traced out by the motion of the point Q will be a Parabola.



PROPERTY 10.

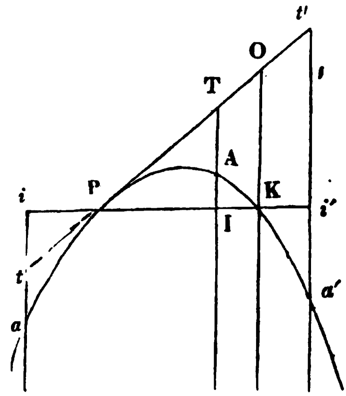
(28.) If QY be a tangent at Q, and VP be produced to meet it in t, then Vt is bisected in P.

Produce QV to q, and draw qY parallel to Vt; then by sim. triangles, (since Qq=2QV,) QY will be double of Qt, and qY double of Vt.

By Art. 26, $Pt : qY :: Qt^2 : QY^2 :: 1 : 4$;
$\therefore Pt = \frac{1}{4} qY$.

But $Vt = \frac{1}{2} qY$, $\therefore Pt = \frac{1}{2} Vt$, or V$t$ is bisected in P. That is, if a tangent and ordinate to any diameter be drawn from the same point, their intersections with the diameter and diameter produced will be equidistant from the vertex of that diameter.

The same may be proved of a tangent at q. Therefore the tangents drawn from the two extremities of any double ordinate intersect the diameter to which that double ordinate belongs in the same point.



(28.a.) This proposition may be thus generalized. Let PO be any tangent, and PK any line in the Parabola, drawn from the point of contact, and meeting the curve in K. Let AT be any diameter produced to meet the tangent in T, and cutting the line PK in I. Then,

$$AT : AI :: PI : IK.$$

For (26.) $AT : KO :: TP^2 : PO^2 ::$ (sim. tri.) $IT^2 : KO^2$.

Hence (Euc. Def. 11. 5.) IT is a mean poportional between AT and KO; or

$$AT : IT :: IT : KO :: \text{(sim. tri.) } PI : PK.$$

Inverted division, $AT : AI :: PI : IK$.

That is, if from any point in the curve, there be drawn a tangent, and also a line to meet the curve in some other place; and if any diameter, intercepted by this line be produced to meet the tangent; then will the curve divide the diameter in the same ratio in which the diameter divides the line.

The same demonstration, very slightly modified, will apply to diameters intersecting the line PK, produced either way, *without* the section, as at, and $a't'$, of which it may be proved that

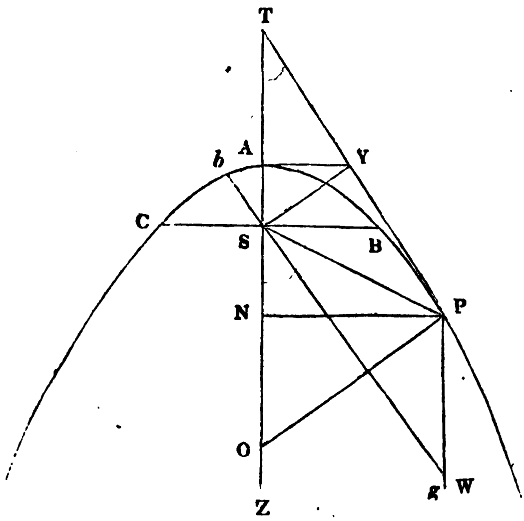
$$at : ai :: Pi : iK,$$

and $$a't' : a'i' :: Pi' : i'K.$$

Property 11.

(29.) Let fall SY perpendicular upon PT, and let AY, be the vertical tangent. AY and SY intersect PT in the same point Y.

Since (17.) ST = SP, and SY is perpendicular to PT, it will divide the triangle PST into two equal triangles; consequently TY = YP; but (18.) TA is also equal to AN; ∴ TY : YP :: TA :: AN;



hence, (Euc. 6. 2.) AY is parallel to PN, and consequently perpendicular to the line AZ. That is, the vertical tangent intersects any other tangent, in the point where a perpendicular from the focus upon that tangent intersects it.

(30.) Cor. Since the normal PO is perpendicular to PT, it is parallel to SY, ∴ TS : SO :: TY : YP; but TY = YP, ∴ TS = SO. Hence, (since SP = TS = SO,) if a circle be described with center S at the distance SP, it will pass through the points P, T, and O; and the ∠ OSP at the center will be double of the angle OTP at the circumference.

PROPERTY 12.

(31.) PO is a mean proportional between BS and *bg*.

Since PY = YT, OS = ST = SP, and TO = 2SP = *bg* (24.) Also (21.) ON = BS.

But (Euc. 8. 6. Cor.) ON : OP :: OP : OT

∴ BS : OP :: OP : *bg*;

that is, the normal is a mean proportional between the semiparameters of the axis and the diameter at the point of contact.

(32.) Cor. 1. SA, SY and ST$=$SP, are severally halves of ON, OP and OT. $\therefore$ SA : SY :: SY : SP; and $SY^2 = SA.SP$; or $SY = \sqrt{(SA.SP)}$ and as SA is constant, $SY \propto \sqrt{(SP.)}$

(32.*a*.) Cor. 2. Since $OP^2 = BS.bg$, and BS is constant; $OP^2 \propto bg \propto 2bg$. And $OP \propto \sqrt{(2bg.)}$ That is, the normal varies as the square root of the parameter to the diameter at the point of contact.

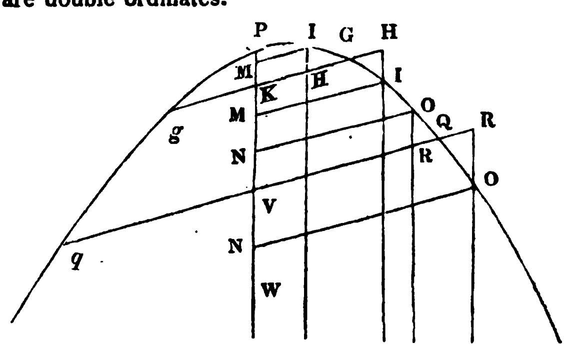
(32.*b*.) Cor. 3. Since SO$=$SP, $\angle$SPO$=\angle$SOP$=\angle$OPg; or, the normal bisects the angle made by the diameter at the point of contact, with the line drawn from that point to the focus.

(32.*c*.) Scholium. In optics, the angle made by a ray of light incident upon a reflecting surface, with a perpendicular to that surface, is called the *angle of incidence;* and the angle made by a reflected ray with the same perpendicular, is called the *angle of reflection.* It is a general law that the angles of incidence and reflection are *equal.* Hence, if CAP represents a concave parabolic mirror, a ray of light falling upon it in the direction gP, will be reflected to S. The same would be true of all rays parallel to gP. Hence the point S, in which all the rays would intersect each other, is called the *focus.*

Property A.

(32.*d*.) Let IH, OR be any two diameters intersected by the parallels gG, qQ, in H, R. Then, IH : OR :: GH.Hg : QR.Rq, whether the points H and R, be within or without the section.

Let P represent the parameter to the diameter PW, of which Gg and Qq are double ordinates.



Then (25.) $P.PV = QV^2$.

And $P.PN = ON^2 = VR^2$.

Taking the diff.

$$P.NV (= P.OR) = QV^2 \backsim VR^2 = (\text{Euc. 5. 2. Cor.}) RQ.Rq.$$

In like manner $P.IH = GH.Hg$.

Hence, $GH.Hg : QR.Rq :: P.IH : P.OR :: IH : OR$;

That is, the parts of all diameters, intercepted by lines parallel to each other, whether within or without the Parabola, are as the rectangles of the corresponding segments of the lines.

Property B.

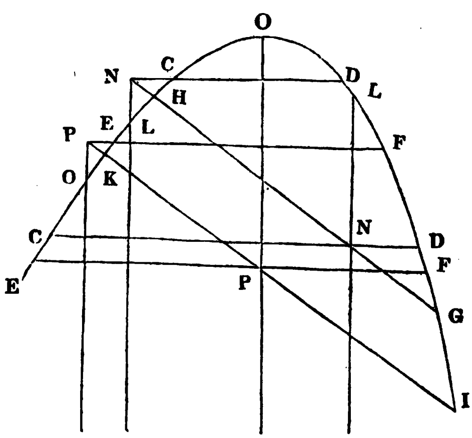
(32.*e.*) Let the parallels CD, EF intersect the parallels GH, IK, in the points N, P. Then

$$CN.ND : HN.NG :: EP.PF : KP.PI.$$

For (32.*d.*) $CN.ND : EP.PF :: LN : OP$

And $HN.NG : KP.PI :: LN : OP$.

$$\therefore CN.ND : EP.PF :: HN.NG : KP.PI;$$



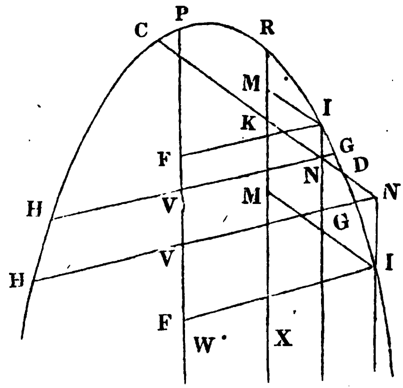
or (Euc. 16. 5.) $CN.ND : HN.NG :: EP.PF : KP.PI$;

Or, the rectangles of the corresponding segments, into which parallel lines in a Parabola divide each other, have to each other a constant ratio.

PROPERTY C.

(32.*f*.) Let RX and PW be the diameters to which CD and HG are double ordinates. Let P represent the parameter of PW, and P′ that of RX.

Then $CN.ND : HN.NG :: P' : P.$



By reasoning like that employed in Prop. A, it may be shown that, $CN.ND = P'.IN$ and $HN.NG = P.IN$

$\therefore CN.ND : HN.NG :: P'.IN : P.IN :: P' : P.$

This proposition may be thus enunciated.

If any two straight lines, which meet the curve in two points intersect each other, the rectangles of their corresponding segments will be as the parameters of the diameters, to which those lines are double ordinates.

The last two propositions, like Property A, are applicable to lines both within and without the section, and the diagrams are lettered in such a manner that the demonstration may apply to either case.

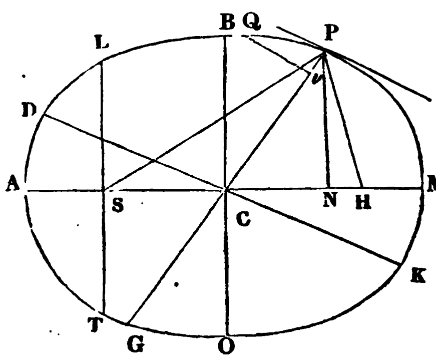
CHAPTER III.

ON THE ELLIPSE.

V.

DEFINITIONS.

(33.) LET APMO be an Ellipse generated by the revolution of the lines SP, HP, about the fixed points S, H, according to the law prescribed in Art. 8.; then the line AM, which passes through the two foci S and H, is called the Axis Major; and if through the center C a line BCO be drawn at right angles to AM, it is called the Axis Minor of the Ellipse.



(34.) From any point P let fall the perpendicular PN upon the axis major AM, and through the focus S draw the straight line LST parallel to it. PN is then called the ordinate to the axis; AN, NM, the Abscissas; and the line LST is called the latus-rectum, or the Parameter to the Axis.

(35.) Draw any line PCG through the center, and another line DCK parallel to a tangent at P; draw also Q*v* parallel to DCK. PCG is then called a Diameter, and DCK the Conjugate diameter to PCG; Q*v* is called an Ordinate to the diameter PCG, and P*v*, *v*G, the Abscissas.

VI.

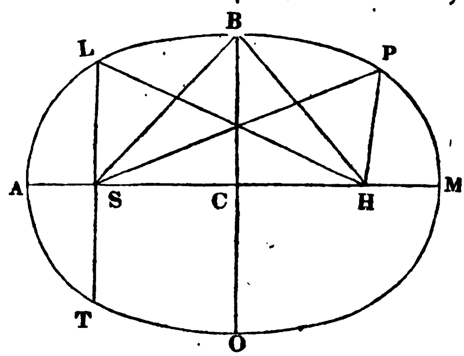
On the Properties of the Ellipse.

PROPERTY 1.

(36.) If SB, HB, are drawn from the foci to the extremity of the axis minor, then SB, HB, are each equal to AC.

Since SC=CH, and BC is common to the two right angled triangles BCS, BCH, SB must be equal to BH; $\therefore$ SB+BH=2SB or 2BH.

Again, by Sect. 2. Art. 8. SP+PH=AM=2AC; and when P



comes to B, SB+BH=2AC; hence 2SB or 2BH=2AC; $\therefore$ SB or BH=AC.

That is, the distance from either focus to the extremity of the axis minor is equal to the semi-axis major.

PROPERTY 2.

(37.) $MS \times SA = BC^2$

For $BC^2 = SB^2 - SC^2$ (Euc. 47. 1.)

$= AC^2 - SC^2$ (by Prop. 1.)

$= (AC + SC) \times (AC - SC)$

$= (CM + SC) \times (AC - SC)$ (for $CM = AC$)

$= MS \times SA$.

That is, the rectangle of the focal distances from the vertices is equal to the square of the semi-axis minor.

(38.) Cor. In the same manner, it might be shown, that $AH \times HM = BC^2$.

Property 3.

(39.) The latus-rectum LST is a third proportional to the major and minor axes.

For
$SL + LH = 2AC$ (by const.)
$\therefore LH = 2AC - SL$,
and $LH^2 = 4AC^2 - 4AC \times SL + SL^2$,

Again,
$LH^2 = SL^2 + SH^2$ (Euc. 47. 1.)
$= SL^2 + 4SC^2$ (for $SH = 2SC$)
$= SL^2 + 4(SB^2 - BC^2)$
$= SL^2 + 4(CA^2 - BC^2)$.

Hence $4AC^2 - 4AC \times SL + SL^2 = SL^2 + 4AC^2 - 4BC^2$;
$$\therefore 4AC \times SL = 4BC^2.$$

And putting this equation into a proportion, we have $\left\{\begin{array}{l} 2AC : 2BC :: 2BC : 2SL, \\ \text{or } AM : BO :: BO : LT. \end{array}\right.$

Therefore the latus-rectum is a third proportional to the major and minor axes.

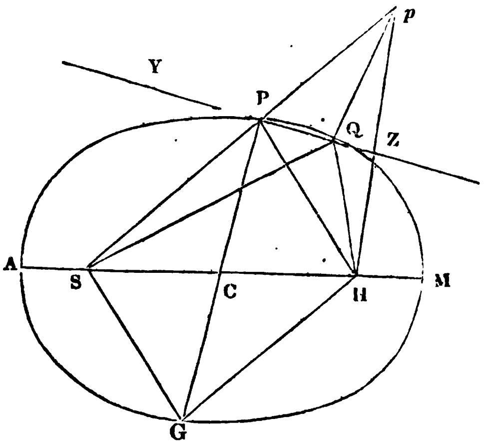
(39.*a*.) $AS.SL = \frac{1}{4}LT.AM$. SM
For (39) $BO^2 = LT.AM$.
$\therefore \frac{1}{4}BO^2 (= BC^2) = (37) AS.SM = \frac{1}{4}LT.AM$.

Property 4.

(40.) Produce SP to p; then if YZ bisects the angle HPp, it will be a tangent to the Ellipse in P. (Fig. in page 29.)

For if YZ does not touch the ellipse, let it cut it in Q; take $Pp = PH$, and join pH, QS, QH, and Qp. Since $Pp = PH$, PZ common, and $\angle pPZ = HPZ$, the side pZ will be equal to ZH; and the $\angle$s PZp, PZH, will be equal and consequently right $\angle$s. Again, since $pZ = ZH$, ZQ common, and $\angle$s QZp, QZH right $\angle$s, the side Qp is equal to the side QH.

Now (by Euc. 20. 1.) $SQ + Qp$ is greater than Sp or $SP + Pp$ or $SP + PH$; but $QH = Qp$; therefore $SQ + QH$ is greater than $SP + PH$; but if Q is a point in the curve, $SQ + QH$ must be equal



to SP+PH; Q therefore is not a point in the curve. In the same manner it might be proved that YZ does not meet the curve in any other point on either side of P, it must therefore be a tangent at P.

Hence, if from the foci two straight lines be drawn to any point in the curve, the straight line bisecting the angle adjacent to that contained by these lines, is a tangent.

(41.) Cor. 1. It follows, from the above that the $\angle$SPY= $\angle$HPZ; for $\angle$SPY=vertical$\angle p$PZ; but pPZ=HPZ; ∴SPY= HPZ; and this is a distinguishing property of the ellipse; viz. That lines drawn from the foci to any point in the curve make equal angles with the tangent at that point.

(41.*a*.) Hence, also, (see Art. 32.*c*.) if rays of light proceed from one focus of a concave ellipsoidal mirror, they will be reflected by the mirror into the other focus.

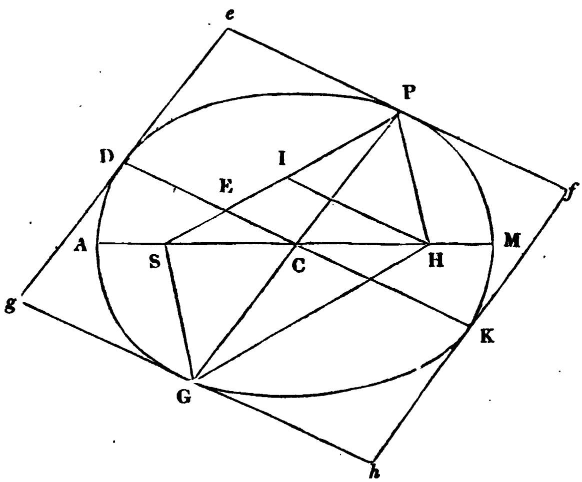
(42.) Cor. 2. When P comes to A or M, the angle HPp becomes equal to two right angles; at A or M, therefore, the tangent is perpendicular to the axis AM.

Property 5.

(43.) If tangents be drawn at the extremities of any diameter of an Ellipse, they will be parallel to each other.

Complete the parallelogram SPHG, of which SP, PH are two sides, and join PG; then since the opposite sides of parallelograms are equal to each other, SG+GH is equal to SP+PH, and consequently G is a point in the Ellipse; and since the diagonals of parallelograms bisect each other, SH is bisected in C; therefore C is the center of the Ellipse, and PG a diameter (35).

Now let the tangents *ef*, *gh* be drawn at the extremities of the diameter PG; then by Art. 41. the $\angle$ SP*e*=$\angle$ HP*f*; but SP*e*+HP*f* is the supplement of $\angle$ SPH; $\therefore$ SP*e*=$\frac{1}{2}$ supplement of SPH. For



the same reason, the $\angle$ HG*h*=$\frac{1}{2}$supplement of $\angle$ SGH; but the $\angle$s SGH, SPH are equal, being opposite $\angle$s of a parallelogram; hence the $\angle$ SP*e*=$\angle$ HG*h*. Again, since SP is parallel to GH, the $\angle$ SPG=$\angle$ PGH; therefore SP*e*+SPG=HG*h*+PGH, or GP*e*= PG*h*, and consequently *ef* is parallel to *gh*. Therefore, if tangents, &c.

(44.) Hence, if tangents be drawn at the extremities of any two diameters of an Ellipse, they will form a parallelogram (*e g h f*.)

PROPERTY 6.

(45.) If SP intersects the semi-conjugate diameter (CD) in E, then PE is equal to the semi-major axis (AC).

Draw HI parallel to CD or ef, then the $\angle$PIH=alternate $\angle$SPe, and $\angle$PHI=alternate $\angle$HPf; but $\angle$SPe=$\angle$HPf, $\therefore$ $\angle$PIH=$\angle$PHI, and consequently PI=PH. Again, since CE is parallel to HI, and SC=CH, SE must be equal to EI.

$$\text{Hence PI}=\frac{\text{IP}+\text{PH}}{2},$$
$$\text{EI}=\frac{\text{SI}}{2};$$
$$\therefore (\text{PI}+\text{EI}) \text{ or PE}=\frac{\text{SI}+\text{IP}+\text{PH}}{2},$$
$$=\frac{\text{SP}+\text{PH}}{2}=\frac{2\text{AC}}{2}=\text{AC}.$$

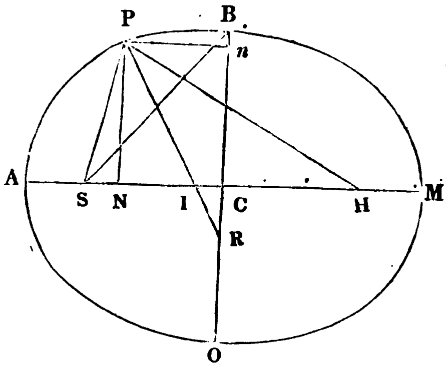
Therefore, if from the extremity of any diameter, a line be drawn to the focus, meeting the conjugate diameter, the part intercepted by the conjugate will be equal to the semi-major axis.

PROPERTY 7.

(46.) If the ordinate PN be drawn to the major axis, then AN $\times$ NM : PN^2 :: AC^2 : BC^2 :: AM^2 : BO^2. (Fig. on p. 32.)

$$\begin{array}{r} \text{Let} \\ \text{AC or CM}=a, \\ \text{SC or CH}=b, \\ \text{CN}=x, \\ \text{and PN}=y; \end{array}\left\{\begin{array}{l} \\ \text{Then, by Art. 8, page 7,} \\ (a^2-b^2)\times(a-x)\times(a+x)=a^2y^2, \\ \text{or } (\text{AC}^2-\text{SC}^2)\times\text{AN}\times\text{NM}=\text{AC}^2\times\text{PN}^2. \\ \text{But AC}^2-\text{SC}^2=\text{SB}^2-\text{SC}^2=\text{BC}^2; \end{array}\right.$$
$$\therefore \text{BC}^2\times\text{AN}\times\text{NM}=\text{AC}^2\times\text{PN}^2,$$
$$\text{and AN}\times\text{NM} : \text{PN}^2 :: \text{AC}^2 : \text{BC}^2 :: \text{AM}^2 : \text{BO}^2.$$

That is, as the square of the axis major is to the square of the axis minor, so are the rectangles of the abscissas of the former, to the squares of their ordinates.



(47.) Cor. Since $AN \times NM = (AC - CN) \times (AC + CN) = AC^2 - CN^2$; $\therefore AC^2 - CN^2 : PN^2 :: AC^2 : BC^2$.

Property A.

(47.*a*.) If from P, the line $PR = AC$, be drawn to BO, then $PI = BC$.

For (sim. tri.) $Rn^2 (PR^2 - Pn^2) : PN^2 :: PR^2 : PI^2$.

That is, $AC^2 - CN^2 : PN^2 :: AC^2 : PI^2$.

But (47) $AC^2 - CN^2 : PN^2 :: AC^2 : BC^2$.

$\therefore PI^2 = BC^2$ and $PI = BC$.

Or, if from any point in the Ellipse, a line be drawn to the minor axis, equal to the semi-major, the part intercepted between that point and the major is equal to the semi-minor axis.

Property 8.

(48.) If the ordinate Pn be drawn to the minor axis, then $Bn \times nO : Pn^2 :: BC^2 : AC^2$.

In this case, $Pn = CN$, and $Cn = PN$; $\therefore AN \times NM = AC^2 - Pn^2$, (47); hence, by substitution in Art. 47, we have,

$$AC^2 - Pn^2 : Cn^2 :: AC^2 : BC^2,$$
$$\therefore AC^2 : AC^2 - Pn^2 :: BC^2 : Cn^2;$$
$$\text{and } Pn^2 : AC^2 :: BC^2 - Cn^2 : BC^2,*$$

---

* For $AC^2 : AC^2 - (AC^2 - Pn^2)$ or $Pn^2 :: BC^2 : BC^2 - Cn^2$, $\therefore$ (invertendo) $Pn^2 : AC^2 :: BC^2 - Cn^2 : BC^2$.

$$::(BC-Cn)\times(BC+Cn) : BC^2,$$
$$:: \quad Bn\times nO \quad : BC^2,$$

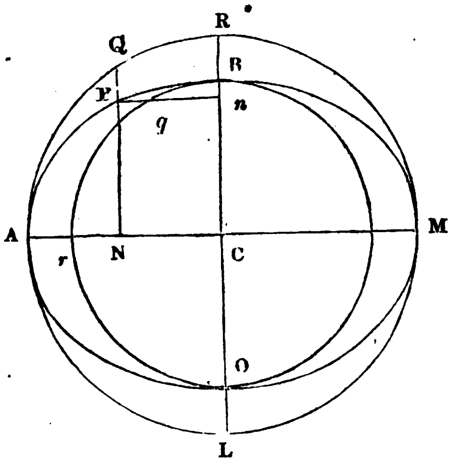
Hence $Bn\times nO : Pn^2 :: BC^2 : AC^2$.

That is, as the square of the axis minor is to the square of the axis major, so are the rectangles of the abscissas of the former, to the squares of their ordinates.

(49.) Cor. Since $Bn\times nO = BC^2 - Cn^2$, we have $BC^2 - Cn^2 : Pn^2 :: BC^2 : AC^2$.

Property 9.

(50.) Describe the circle ARML upon the major axis AM, and draw an ordinate QPN cutting the ellipse in P; then $QN : PN :: AC : BC$.



By Art. 46. $AN\times NM : PN^2 :: AC^2 : BC^2$.
But by Prop. of circle, $AN\times NM = QN^2$;

$$\therefore QN^2 : PN^2 :: AC^2 : BC^2,$$
$$\text{and } QN : PN :: AC : BC.$$

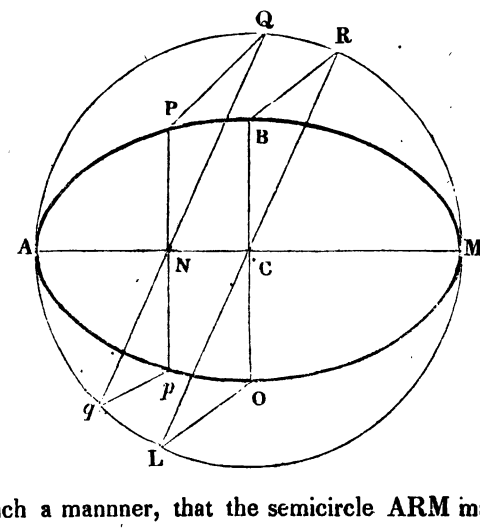
In like manner, it may be shown that $qn : Pn :: BC : AC$. Hence, if a circle be described on either axis, then any ordinate in

the circle, is to the corresponding ordinate in the Ellipse, as the axis of that ordinate is to the other axis.

(51.) Cor. 1. Since RC is equal to AC, QN : PN::RC : BC. Hence it appears, that if upon AM as diameter, a circle be described, and if B be a given point in the line RC; then if the ordinates of this circle are diminished in the given ratio of RC : BC, the curve APBM passing through the extremities of these lesser ordinates, will be an ellipse, whose axis major is to the axis minor in the same given ratio.

Also, if upon BO a circle be described, and if A be a point in C*r*, produced; then if the ordinates of this circle are increased in the given ratio of C*r* : CA, the curve BPAO, passing through the extremities of these greater ordinates will be an Ellipse, whose axis minor is to its axis major in the same given ratio.

(52.) Cor. 2. From hence also it may be shown, that the orthographic projection of a circle upon a plane will be an Ellipse. Suppose the circular plane ARML to be inclined to the plane of this



paper in such a mannner, that the semicircle ARM may be above the paper, and the semicircle ALM below it, and let AM be the common intersection of the two planes. Let the semicircle ARM be

projected downwards upon the plane of the paper, by drawing perpendiculars QP, RB, from each point of the circle, and let the semicircle ALM be projected upwards, by drawing the perpendiculars qp, LO, &c.; then the curve ABMO, marked out by this projection, will be an ellipse. For draw QN, RC, at right angles to AM, and join PN, BC; then the angles QNP, RCB, will measure the inclination of the planes, and PN, BC will be perpendicular to their common intersection AM. Now QN : PN : : rad. : cos $\angle$ QNP, and RC : BC : : rad. : cos. RCB(=QNP); $\therefore$ QN : PN : : RC or AC : BC; and consequently, the four lines QN, PN, RC, BC, bear the same relation to each other and to AM as they did in Cor. 1; hence P, B, &c. are points in an ellipse. In the same manner it may be proved, that the semicircle ALM is projected into a semi-ellipse AOM; and thus the whole circle ARML is projected into an ellipse ABMO, whose axis major is AM.

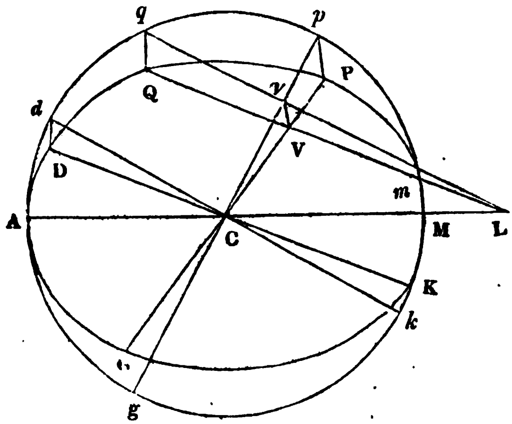
This proposition is likewise manifestly true, when the plane of projection does not cut the circle, or cuts it unequally.

Property 10.

(53.) Let PCG be any diameter of an ellipse, and DCK its conjugate diameter; draw the ordinate QV, then

$$PG^2 : DK^2 : : PV.VG : QV^2.$$

Let the circle $AqMg$ be projected into the ellipse AQMG, according to the principles just now laid down, and let the diameter pCg of the circle be projected into the diameter PCG of the ellipse. Draw the diameter dCk, at right angles to pCg, and qv parallel to dCk, and let dCk, qv be projected into DCK, QV; then since parallel lines are projected into parallel lines, QV will be parallel to DCK. Now it is evident that a tangent to the circle at p would be projected into a tangent to the ellipse at P; dCk and qv therefore being parallel to a tangent at p, (for they are both perpendicular to pCg) DCK and QV will both be parallel to a tangent at P; hence DCK is the conjugate diameter, and QV the ordinate, to the diameter PCG. Again, since Qq is parallel to dD (for they are both at right $\angle$s to



the plane of the ellipse,) and QV parallel to DC, the plane of the triangle dDC must be parallel to the plane of the figure QqvV; but qv is parallel to dC; if therefore QV and qv are produced till they meet in L, they will form a triangle QdL, similar to the triangle dDC; and since qL is in the plane of the circle, and QL in the plane of the ellipse, the point L must be in the common intersection (AM produced) of those planes. Now pP, vV being perpendicular to the plane of the ellipse, are parallel to each other, and to the lines Qq, dD; hence it appears that the triangles QqL, VvL, dDC, and the triangles pPC, vVC, are respectively similar; we have then, by property of circle,

$$Cp^2 : Cp^2 - Cv^2 (pv.vg \text{ Euc. 5. 2. Cor.}) :: Cd^2 : qv^2.$$

But, (sim. tri.) $Cp^2 : Cp^2 - Cv^2 :: CP^2 : CP^2 - CV^2 (PV.VG)$

And $Cd^2 : qv^2 :: CD^2 : QV^2.$

$$\therefore CP^2 : PV.VG :: CD^2 : QV^2;$$

Or, $CP^2 : CD^2 :: PG^2 : DK^2 :: PV.VG : QV^2.$

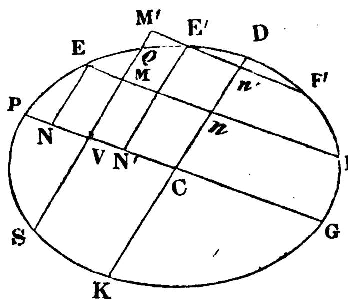
The same demonstration is, obviously, applicable to Vm, and CK. Consequently, as the square of any diameter is to the square of its conjugate, so are the rectangles of its abscissas to the squares of their ordinates.

(54.) Cor. Since any diameter in the Ellipse is the projection of a corresponding diameter in the circle, and since all the diameters of the circle are bisected in the center; it follows that all diameters of the Ellipse are bisected in the center. For similar reasons, every diameter in the Ellipse bisects its double ordinates, or lines drawn in the Ellipse, parallel to the tangent at its vertex.

Property B.

(54.*a*.) Let PG, DK be any two conjugate diameters, and EF, QS any lines parallel to PG, DK, intersecting each other in M. Then $PG^2 : DK^2 :: EM.MF : QM.MS$.

Draw the ordinate EN.



Then (53.) $PG^2 : DK^2 :: PN.NG(CP^2-CN^2) : EN^2$.

Also $PG^2 : DK^2 :: CP^2-CV^2 : QV^2$.

$\therefore$ (Euc. 19. 5.) $PG^2 : DK^2 :: CN^2-CV^2(=En^2-Mn^2)$

$: QV^2-EN^2(=MV^2)$.

That is, (Euc. 5. 2. Cor.) $PG^2 : DK^2 :: EM.MF : QM.MS$.

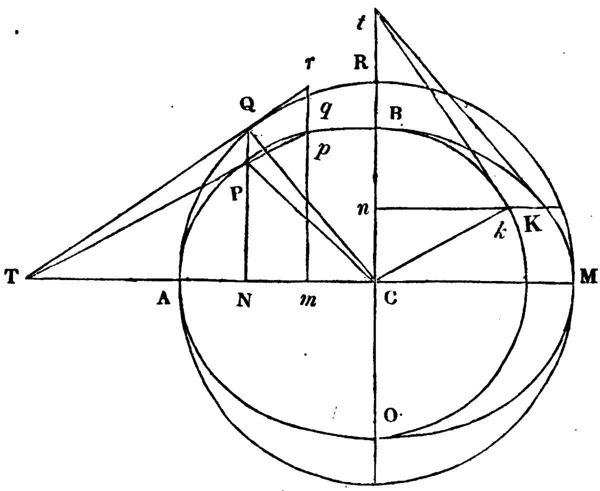
The same demonstration is applicable, (with a single change of sign) to E′F′, which intersects QV in M′ without the Ellipse. Wherefore, if straight lines in the Ellipse parallel to two conjugate diameters intersect each other either within or without the Ellipse, the rectangles of their corresponding segments are to each other as the squares of the diameters to which they are parallel.

(54.*b*.) Cor. $PG^2 : DK^2$ or $PC^2 : DC^2 :: En^2-Mn^2 : QV^2-MV^2$.

Property 11.

(55.) If QT, PT, are tangents to the circle and ellipse in the points Q and P, they will meet in the axis produced at T; and CA will be a mean proportional between CN and CT. And if Kt, kt, are tangents to the points K and k, BC will be a mean proportional between Cn and Ct.

Let QT be a tangent to the circle in Q, and join TP. If TP does not touch the ellipse, let it cut it in P, p; and through p draw the ordinate $mpqr$, meeting TQ produced in r.



By similar △s, TNP, Tmp; TNQ, Tmr; we have,

TN : Tm: : PN : pm,

and

TN : Tm: : QN : rm;

∴ PN : pm: : QN : $rm=\frac{pm \times \text{QN}}{\text{PN}}$; but by Art. 50, the ordinates of the circle and ellipse are to each other in a given ratio; therefore QN : PN: : qm : pm, or $qm=\frac{pm \times \text{QN}}{\text{PN}}$. Hence $rm=qm$, which is impossible; ∴ TP does not cut the ellipse, and consequently

is a tangent to it in P; and since TQ touches the circle in Q, CQT is a right-angled triangle, $\therefore$ (Euc. 6. 8.) CN : CQ : : CQ : CT; but CQ=CA, hence CN : CA : : CA : CT, and $CN \times CT = CA^2$.

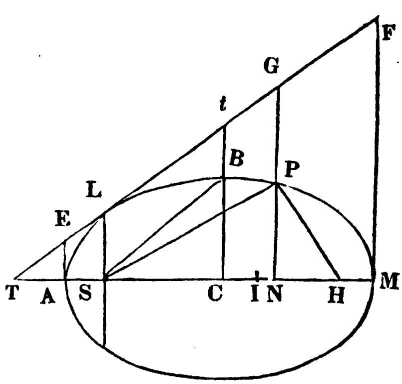
(56.) Upon the axis minor BO describe the circle BkQ; draw the ordinate Kkn, and join Ck. By Art. 48, we have $Bn \times nO$: Kn^2 : : BC^2 : AC^2; by the same process, therefore, as that in Art. 50, it may be proved that Kn is to kn in a given ratio. Draw Kt, kt, to the ellipse and the circle; then proceeding in the same manner as in the former part of this demonstration, we might show that they will meet in the minor axis produced. Since kt touches the circle, Ckt is a right-angled triangle, $\therefore$ Cn : Ck : : Ck : Ct; but Ck=BC, $\therefore$ Cn : BC : : BC : Ct, or $Cn \times Ct = BC^2$.

Hence, if a tangent and an ordinate to either of the axes be drawn to any point of the Ellipse, meeting that axis and axis produced, then the semi-axis is a mean proportional between the distances of the two intersections from the center.

(57) Cor. In the right-angled triangle CQT, (Euc. 8. 6,) CN : QN : : QN : : NT, $\therefore$ $CN \times NT = QN^2 = CQ^2 - CN^2 = AC^2 - CN^2$. For the same reason, $Cn \times nt = BC^2 - Cn^2$.

Property C.

(57.a.) Let TLG be the focal tangent, or the tangent drawn at



the extremity of SL, the ordinate from the focus. Let NPG be any ordinate, produced to meet the tangent TLG. Then SP=NG.

If AI be taken equal to SP, IM=HP.

$\therefore$ SP=CA+CI, and HP=CA−CI

(Euc. 47. 1.) $(SP^2)(CA+CI)^2=PN^2+(CS+CN)^2(NS^2)$.

And (HP^2) $(CA-CI)^2=PN^2+(CS-CN)^2(HN^2)$.

Expand and subtract 4CA.CI=4CS.CN, and CA.CI=CS.CN.

$\therefore$ CN : CI(SP−CA)::CA : CS::CT : CA (55.)

$\therefore$ CN+CT(TN) : SP::CT : CA.

Again, (55.) CS.CT=AC^2, and CS.ST=$CA^2-CS^2=BC^2$ (36, and Euc. 47. 1.) $\therefore$ CS.CT::CS.ST or CT : ST::AC^2 : BC^2::AC : SL(39.)

$\therefore$ ST : SL::CT : AC::TN : SP.

But (sim. tri.) ST : SL::TN : NG.

$\therefore$ SP=NG;

That is, the distance from the focus to any point of the curve is equal to the ordinate to that point, produced to meet the focal tangent.

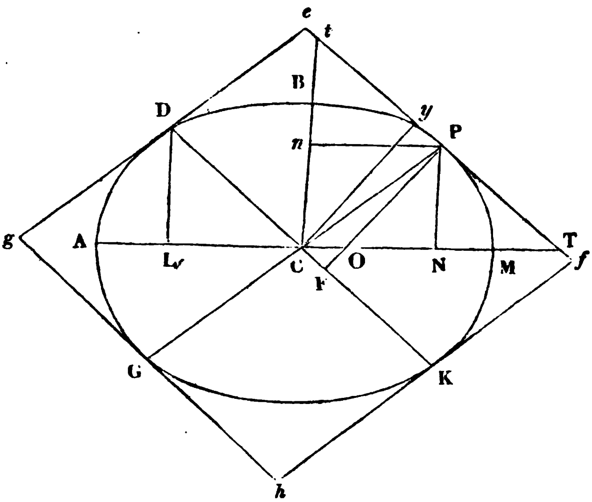
Cor. 1. AS=AE, SM=MF, and Ct=AC.

Cor. 2. Hence, also, since TN : NG is a constant ratio, TN : SP is a constant ratio. Therefore, if a line be drawn through T perpendicular to AC produced, the distance of the point P from this line, (=TN) is in a constant ratio to SP, the distance of the same point from the focus. This ratio, being (by demonstration above,) =CT : CA is a ratio of greater inequallity. This perpendicular is the directrix of the Ellipse. (See Art. 138, *et seq.*)

Property 12.

(58.) If PCG, DCK, be conjugate diameters of the ellipse, and PF perpendicular to CK, then PO×PF=BC^2.

Draw Cy parallel to PF. Then because PO is parallel to Cy, and Ct parallel to PN, and the $\angle$s Cyt, PNO right $\angle$s, the triangles tCy, PON, are similar; $\therefore Ct : Cy :: PO : PN$; but $Cy = PF$, and $PN = Cn$, (being opposite sides of parallelograms); $\therefore Ct : PF ::$



$PO : Cn$, or $PF \times PO = Cn \times Ct = BC^2$ by Art. 56. That is, if from the extremity of any diameter, a perpendicular be drawn to its conjugate; then the rectangle of that perpendicular and the part of it intercepted by the axis major will be equal to the square of the semi-axis minor.

Property 13.

(59.) Draw the ordinates DL, PN, to the major axis, then $CN^2 + CL^2 = AC^2$, and $PN^2 + DL^2 = BC^2$.

By Art. 47,

$$AC^2 - CL^2 \ldots\ldots\ldots\ldots : DL^2 :: AC^2 : BC^2, \text{(A)}$$

$$\text{and } AC^2 - CN^2 \left\{ \begin{matrix} CN \times NT \\ \text{Art. 57.} \end{matrix} \right\} : PN^2 :: AC^2 : BC^2;$$

$$\therefore AC^2 - CL^2 : DL^2 :: CN \times NT : PN^2,$$

$$\text{or } AC^2 - CL^2 : CN \times NT :: DL^2 : PN^2,$$

$$:: CL^2 : NT^2 \text{ by sim. } \triangle\text{s, } \left\{ \text{DCL, PTN} \right\}$$

Hence $AC^2 - CL^2 = \frac{CN \times NT \times CL^2}{NT^2} = \frac{CN \times CL^2}{NT}$.

From which we have,

$$CL^2 : AC^2 - CL^2 :: NT : CN,$$

and *comp^{do}*, $AC^2 : AC^2 - CL^2 :: CT : CN,$

$$:: CT \times CN : CN^2,$$
$$:: AC^2 : CN^2 \text{ (55.)}$$

Hence $AC^2 - CL^2 = CN^2$, or $CN^2 + CL^2 = AC^2$.

(60.) Since $AC^2 - CL^2 = CN^2$, by substitution in proportion (A) we have $CN^2 : DL^2 :: AC^2 : BC^2$; but $AC^2 - CN^2 : PN^2 :: AC^2 : BC^2 \therefore CN^2 : DL^2 :: AC^2 - CN^2 : PN^2$; And alternately,

$$CN^2 : AC^2 - CN^2 :: DL^2 : PN^2,$$

Comp^{do}, $CA^2 : AC^2 - CN^2 :: DL^2 + PN^2 : PN^2$;

$$\therefore AC^2 : DL^2 + PN^2 :: AC^2 - CN^2 : PN^2 :: AC^2 : BC^2.$$

Hence $DL^2 + PN^2 = BC^2$.

Hence if ordinates to either axis be drawn from the extremities of any two conjugate diameters, the sum of their squares will be equal to the square of half the other axis.

Property 14.

(61.) $PC^2 + CD^2 = AC^2 + BC^2$, PG and DK being conjugate diameters. (See last Fig.)

For by Prop. 13. $CN^2 + CL^2 = AC^2$,

and $PN^2 + DL^2 = BC^2$;

$$\therefore CN^2 + PN^2 + CL^2 + DL^2 = AC^2 + BC^2,$$

or $CP^2 + CD^2 = AC^2 + BC^2$.

Therefore the sum of the squares of any two semi-conjugate diameters is equal to the sum of the squares of the semi-axes.

Property 15.

(62.) $CD \times PF = AC \times BC$. (See last Fig.)

In case 2. of Prop. 13. it was proved that

$$CN^2 : DL^2 :: AC^2 : BC^2;$$
$$\therefore CN : DL :: AC : BC,$$
$$\text{and } CN : AC :: DL : BC.$$

By similar $\triangle$s, TCy, DCL, $CT : Cy(PF) :: CD : DL$.

$$\text{Hence we have, } CN : AC :: DL : BC,$$
$$\text{and } CT : PF :: CD : DL,$$
$$\therefore CN \times CT(AC^2) : AC \times PF :: CD : BC,$$
$$\text{or } AC : PF :: CD : BC.$$
$$\therefore CD \times PF = AC \times BC.$$

That is, if from the extremity of any diameter, a perpendicular be drawn to its conjugate, the rectangle of that perpendicular and the semi-conjugate, is equal to the rectangle of the semi-axes.

(63.) Cor. From this it appears, that all the parallelograms circumscribing the ellipse, and having their sides drawn through the extremities of any diameter and its conjugate, are equal to each other and to the parallelogram described about the major and minor axes. For the parallelogram *eghf* described about the conjugate diameters PCG, DCK, is equal to four times $eDCP = 4CD \times PF = 4AC \times BC =$ right-angled parallelogram whose sides are 2AC and 2CB= parallelogram described about the major and minor axes.

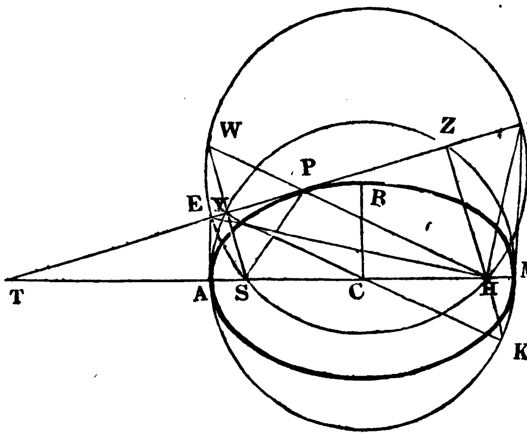
Property 16.

(64.) If SY, HZ, be drawn from the foci perpendicular upon the tangent YZ, the points Y, Z, will be in a circle described upon the major axis AM. (Fig. in p. 44.)

Join YC, produce HP to W, making PW=PS, and join WY.

By Prop. 4. PY bisects the $\angle$ SPW; and since SP=PW, and PY is common, WY will be equal to YS, and $\angle WYP = \angle SYP =$ a right angle; hence WYS is a straight line. Now Since WY=YS,

and SC=CH, CY must be parallel to WH, and ∴ SC : SH :: CY : HW, but $SC=\frac{1}{2}SH$; $\therefore CY=\frac{1}{2}HW=\frac{1}{2}(HP+PS)=\frac{1}{2}AM=AC$; hence Y is a point in the circle whose center is C and radius CA. In the same manner it may be proved that Z is a point in the same circle.



Therefore, if perpendiculars be dropped from the foci upon any tangent to the Ellipse, the intersections of those perpendiculars with the tangent, will be in the circumference of a circle described upon the axis major.

Property 17.

(65.) $SY \times HZ = BC^2$.

Since the $\angle HZP$ is a right angle, it must be in a semi-circle; if ∴ YC and ZH be produced, they will meet in the circumference of the circle at some point, and YK will be a diameter. Hence YC=CK; and as SC=CH, and $\angle SCY=\angle KCH$, the side HK must be equal to SY. But by the property of the circle, (Euc. 3. 35.) $ZH \times HK = AH \times HM = BC^2$ by Art. 38. Hence (since HK=SY) $SY \times HZ = BC^2$.

That is, the rectangle of the perpendiculars from the foci upon any tangent, is equal to the square of the semi-axis minor.

(66.) Cor. Since $\angle$s at Z and Y are right angles, and $\angle SPY = \angle HPZ$, the triangles SPY, HPZ, are similar; hence,

$$SP : SY :: HP : HZ = \frac{SY \times HP}{SP};$$

$$\therefore SY \times HZ = \frac{SY^2 \times HP}{SP},$$

$$\text{or } BC^2 = \frac{SY^2 \times HP}{PS}.$$

From which it follows, that $SY^2 = BC^2 \times \frac{SP}{HP}$;

$\therefore SY = BC \times \sqrt{\left(\frac{SP}{HP}\right)} \propto \sqrt{\left(\frac{SP}{HP}\right)}$, BC being constant.

Property D.

(66.*a.*) Let the vertical tangents AE, MF be drawn; then EA.FM= BC^2 and EF is the diameter of a circle, passing through S and H.

If P coincide with B, then EA and FM each$=$BC, and EA.FM$=BC^2$. But if not, let the tangent PE intersect the axis in T.

Then (sim. tri.) EA : SY :: TA : TY,
and HZ : FM :: TZ : TM.

But (Euc. 36. 3. Cor.) TA.TM=TY.TZ.
or TA : TY :: TZ : TM,
$\therefore$ EA : SY :: HZ : FM,
and EA.FM=SY.HZ=(65.)BC^2.

Again, (38.) AH.HM$=BC^2=$EA.FM,
$\therefore$ AH : EA :: FM : HM.

Hence (Euc. 6. 6.) the triangles EAH and HFM are similar, and $\angle EHA = \angle HFM$ and $\angle FHM = \angle AEH$. Whence $\angle EHF$ is a right angle, and a circle described on EF will pass through H. The same may also be shown of S.

Wherefore if tangents be drawn from the vertices to meet any other tangent, the rectangle of the vertical tangents will be equal to the square of the semi-minor axis; and the intercepted part of the other tangent will be the diameter of a circle passing through the foci.

Property 18.

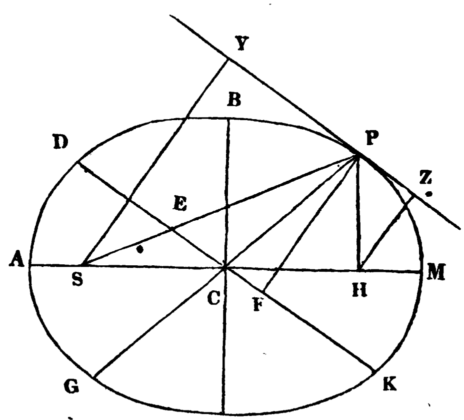
(67.) Draw the conjugate diameters PCG, DCK, then $SP \times HP = CD^2$.

By similar triangles, SPY, HPZ, PEF, we have

$$SP : SY :: PE : PF,$$
$$\text{and } HP : HZ :: PE : PF$$
$$\therefore SP \times HP : SY \times HZ :: PE^2 : PF^2.$$

Now, by Art. 65, $SY \times HZ = BC^2$,
Art. 45. . . $PE^2 = AC^2$;

$$\therefore SP \times HP : BC^2 :: AC^2 : PF^2, \text{ or } SP \times HP = \frac{BC^2 \times AC^2}{PF^2}.$$



But by Art. 62. $CD \times PF = AC \times BC$;

$$\therefore CD = \frac{AC \times BC}{PF} \text{ and } CD^2 = \frac{AC^2 \times BC^2}{PF^2},$$

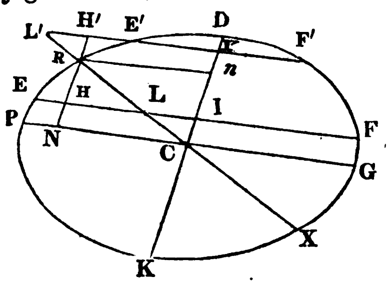
Hence $SP \times HP = CD^2$.

That is, the rectangle contained by the straight lines, drawn from the foci to the extremity of any diameter, is equal to the square of half the conjugate to that diameter.

Property E.

(67.*a*.) Let PG, RX be any two diameters, and let EF, paralle to PG, cut RX in L. Then $PG^2 : RX^2 :: EL.LF : RL.LX$.

Draw DK conjugate to PG, and RN, R*n*, ordinates to PG, DK.



Then $RN = Cn$, $HN = CI$ and $Rn = HI = CN$.

By sim. tri. R*n*C, CLI, $Cn^2 : Cn^2 - CI^2 :: RC^2 : RC^2 - CL^2$.

Also (54.*b*.) $RN^2(Cn^2) : RN^2 - HN^2(Cn^2 - CI^2) :: PC^2 - CN^2 : EI^2 - HI^2(EI^2 - Rn^2)$.

But (sim. tri.) $Cn^2 : Cn^2 - CI^2 :: Rn^2(CN^2) : Rn^2 - LI^2 :: RC^2 : RC^2 - CL^2$

Our proportions then, are

$$Cn^2 : Cn^2 - CI^2 :: PC^2 - CN^2 : EI^2 - Rn^2$$

$$Cn^2 : Cn^2 - CI^2 :: CN^2 : Rn^2 - LI^2 :: RC^2 : RC^2 - CL^2$$

Adding the terms of equal ratios, by Euc. 12. 5. $PC^2 : EI^2 - LI^2 :: RC^2 : RC^2 - CL^2$

Alternation, and Euc. 5. 2, cor. $PC^2 : RC^2 :: EL.LF : RL.LX$,

Or, $PG^2 : RX^2 :: EL.LF : QL.LX$.

The same demonstration, (signs being changed whenever necessary,) is applicable to E′F′, which intersects RX, produced, in L′.

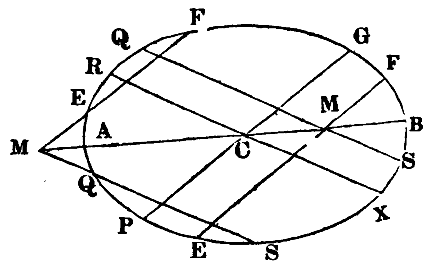
Wherefore, the squares of any two diameters are to each other, as the rectangles of the segments of one of them, are to the rectangles of the corresponding segments of lines parallel to the other; whether the point of intersection be within or without the ellipse.

Property F.

(67.*b*.) Let PG, RX, be any diameters, and let EF, QS, parallel to PG, RX respectively, intersect each other in M.

Then $PG^2 : RX^2 :: EM.MF : QM.MS$.

For, through M draw the diameter AB.



$$\text{Then } AB^2 : PG^2 :: AM.MB : EM.MF.$$
$$\text{Or, } AB^2 : AM.MB :: PG^2 : EM.MF.$$
$$\text{In like manner, } AB^2 : AM.MB :: RX^2 : QM.MS.$$
$$\therefore PG^2 : RX^2 :: EM.MF : QM.MS.$$

Which demonstration is equally applicable to lines intersecting within or without the ellipse.

Wherefore, if straight lines in the ellipse intersect each other, either within or without the curve, the rectangles of their corresponding segments are to each other as the squares of those diameters, to which they are parallel.

Cor. When a line becomes a *tangent*, its *square* corresponds to the *rectangle* in other cases. Therefore the squares of tangents which intersect, are as the squares of the diameters to which they are parallel, and the tangents themselves are as the same diameters.

These are a few of the most useful properties of the Ellipse; a variety of others will be found in the Sixth Chapter, which treats of the analogous properties of the three Conic Sections. We now proceed to the Hyperbola.

CHAPTER IV.

ON THE HYPERBOLA.

The Properties of the Hyperbola may be divided into two classes; in the first class may be placed such properties as are analogous to those of the Ellipse, in the second class such as are derived from its relation to the Asymptote. We shall consider each of these classes separately, beginning with that which contains the properties analogous to those of the Ellipse.

VII.

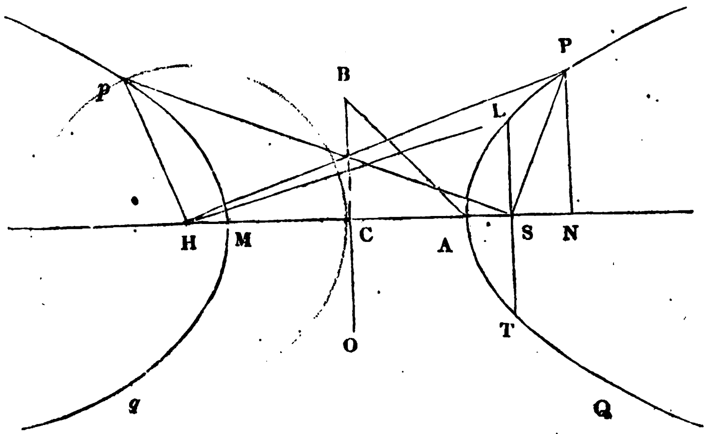
DEFINITIONS.

(68.) Let PAQ be an Hyperbola generated by the revolution of the lines SP, HP about the fixed points S, H, according to the law prescribed in Art. 9. Take HM = AS, and let the lines Sp, Hp, revolve round H, S, according to the same law; then it is evident that the point p will trace out another curve pMq passing through M precisely similar to PAQ. pMq is therefore called the opposite Hyperbola.

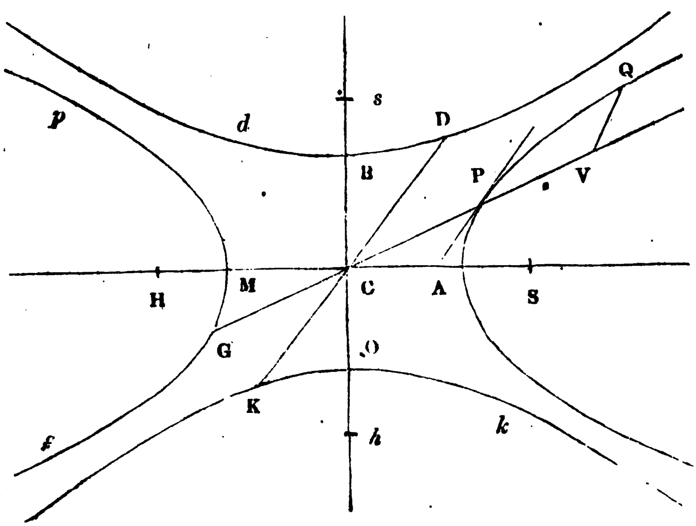
(69.) The point A is called the vertex; and the part AM of the line HS which joins the two foci S and H, is called the Major axis of the Hyperbola.

(70.) If AM be bisected in C, C is called the center; and if through C a line BCO be drawn at right angles to AM, and with center A and radius SC a circle be described cutting BCO in B and O, (in which case $BC^2 = AB^2 - AC^2 = SC^2 - AC^2$), then BCO is called the Minor axis of the Hyperbola.

(71.) From any point P let fall the perpendicular PN upon the axis major MA produced, and through the focus S draw LST par-



allel to it; then PN is called the Ordinate to the axis; AN, NM, the Abscissas; and the line LST is called the Latus-rectum, or the Parameter to the axis.



(72.) Produce BCO both ways; take C*s*, C*h* equal to CS or CH; and with *s*, *h* as foci, BO major axis, and AM conjugate axis,

describe two other hyperbolas dBD, KOk; these are called conjugate Hyperbolas. A figure thus arises consisting of four Hyperbolas, with their vertices A, B, M, O, turned towards each other, of which the opposite parts are similar and equal. If BCO=ACM, then these four Hyperbolas are exactly similar and equal; and in this case the Hyperbolas are said to be Equilateral.

(73.) Any line PCG drawn through the center, and terminated by the opposite hyperbolas, is called a diameter; the line DCK drawn parallel to a tangent at P, and terminated by the conjugate hyperbolas, is called a conjugate diameter to PCG. From any point Q, draw QV parallel to a tangent at P; then QV is called the ordinate to the diameter PCG, and PV, VG the abscissas.

VIII.

Properties of the Hyperbola analogous to those of the Ellipse.

Property 1. (Prop. 2, of Ellipse.)

(74.) $MS \times SA = BC^2$. (See Fig. in page 50.)

By Art. 70. $BC^2 = SC^2 - AC^2$.

$$= (SC + AC) \times (SC - AC).$$
$$= (SC + CM) \times (SC - AC) \text{ (for } CM = AC.)$$
$$= MS \times SA.$$

That is, the rectangle of the focal distances from the vertices, is equal to the square of the semi-axis minor.

Cor. For the same reason, $AH \times HM = BC^2$.

Property 2. (Prop. 3, of Ellipse.)

(75.) The latus-rectum LST is a third proportional to the major and minor axes. (Fig. in page 50.)

For $HL - SL = 2AC$ (by const.)
$\therefore HL = 2AC + SL$,
and $HL^2 = 4AC^2 + 4AC \times SL + SL^2$.

Again,
$HL^2 = SL^2 + SH^2$ (Euc. 47. 1.)
$= SL^2 + 4SC^2 (SH = 2SC)$,
$= SL^2 + 4AB^2$,
$= SL^2 + 4(AC^2 + BC^2)$.

Hence $4AC^2+4AC\times SL+SL^2=SL^2+4AC^2+4BC^2$;

$$\therefore 4AC\times SL=4BC^2,$$

$$\text{and } 2AC : 2BC :: 2BC : 2SL,$$

$$\text{or } AM : BO :: BO : LT.$$

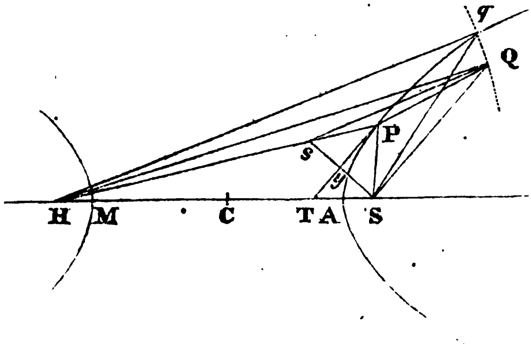
Or, the latus-rectum is a third proportional to the major and minor axes.

(75.*a*.) $AS.SM=\frac{1}{4}LT.AM$. For (75.) $BO^2=LT.AM$.
$\therefore \frac{1}{4}BO^2(=BC^2)=(74.)AS.SM=\frac{1}{4}LT.AM$.

PROPERTY 3. (Prop. 4, of Ellipse.)

(76.) If PyT bisects the angle HPS, it will be a tangent to the Hyperbola in P.

If PT be not a tangent, it must cut the curve in P. Let Q be any point *within the Hyperbola*, in TP produced. Draw Sys at right angles to TP, meeting HP in s. Join HQ, SQ, sQ. With H as center, and HQ radius, describe the circular arc Qq, cutting the curve in q. Join qS, qH. Then, since qS, QS are the bases of the triangles qHS, QHS and $\angle\, q$HS $>$ $\angle$ QHS, qS $>$ QS. (Euc. 24. 1.)



In the right angled triangles SyP, syP, Py is common, and $\angle\, SPy=\angle\, sPy$. $\therefore Sy=sy$, and $Ps=PS$. Also in the right angled triangles QSy, Qsy, $sy=Sy$, and Qy is common, $\therefore Qs=QS$.

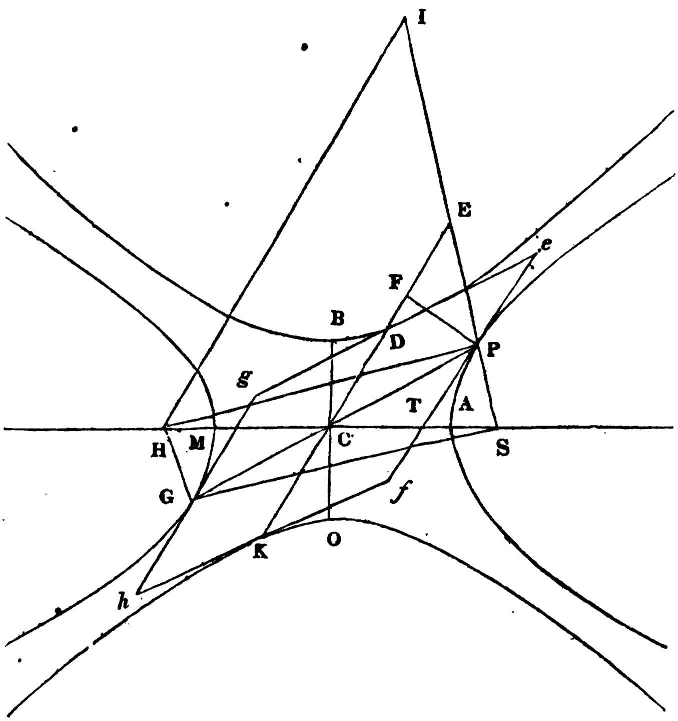
Since $PS = Ps$, $HP - Ps = HP - PS = AM$; i. e. $Hs = AM$. Also, since $QS = Qs$, $HQ - Qs = HQ - QS > Hq - qs = AM$; $\therefore$ $HQ - Qs > Hs$. Or $HQ > Hs + Qs$; which (Euc. 20. 1.) is impossible. Hence TP does not cut the curve, that is, it touches it.

Therefore, if from the foci two straight lines be drawn to any point in the curve, the straight line bisecting the angle contained by these lines, is a tangent.

(77.) Cor. When P comes to A, the $\angle HPS$ = two right angles; therefore a tangent at A is perpendicular to the axis AM.

Property 4. (Prop. 5, of Ellipse.)

(78.) If tangents be drawn at the extremities of any diameter of an Hyperbola, they will be parallel to each other.



Complete the parallelogram SPHG, of which HP, PS are two sides; then since its opposite sides are equal, GS − HG will be equal to HP − PS, and by a process similar to that in the Ellipse (Art. 43.), it may be proved that G is a point in the opposite hyperbola, C the center of the hyperbola, and PG a diameter.

Now in the parallelogram SPHG, the opposite $\angle$HGS$=\angle$HPS; but by Art. 76, PT bisects the $\angle$HPS, and for the same reason G*g* bisects the $\angle$HGS; hence the $\angle g$GP is equal to $\angle$GPT; and therefore *ef* is parallel to *gh*. Therefore, if tangents, &c.

(79.) Hence, (as in the Ellipse), if tangents be drawn at the extremities of any two diameters PCG, DCK, they will form, by their intersection, a parallelogram *eghf*.

PROPERTY 5. (Prop. 6, of Ellipse.)

(80.) If SP and CD be produced till they intersect each other in E, then PE$=$AC.

Draw HI parallel to CDE or *ef*, and produce SE to meet it in I. Since HI is parallel to P*f*, the exterior $\angle$SP$f=$interior $\angle$PIH, and $\angle$HP$f=$alternate $\angle$PHI; but $\angle$SP$f=\angle$HPf, because PT bisects $\angle$HPS; $\therefore$ $\angle$PHI$=\angle$PIH, and consequently PI$=$PH. Again, because CE is parallel to HI, and SC$=$CH, SE must be equal to EI.

$$\text{Hence } \mathrm{PI}=\frac{\mathrm{HP}+\mathrm{PI}}{2},$$

$$\mathrm{EI}=\frac{\mathrm{SI}}{2};$$

$$\therefore (\mathrm{PI}-\mathrm{EI}) \text{ or } \mathrm{PE}=\frac{\mathrm{HP}+\mathrm{PI}-\mathrm{SI}}{2},$$

$$=\frac{\mathrm{HP}-\mathrm{PS}}{2}=\frac{2\mathrm{AC}}{2}=\mathrm{AC}.$$

Hence, if through the extremity of any diameter, a line be drawn from the focus, to meet the conjugate diameter produced, the part intercepted by the conjugate will be equal to the semi-axis major.

PROPERTY 6. (Prop. 7. of Ellipse.)

(81.) If the ordinate PN be drawn to the major axis, then $AN \times NM : PN^2 :: AC^2 : BC^2$. (Fig. in page 56.)

$$\left.\begin{aligned} \text{Let } AC \text{ or } CM &= a, \\ SC \text{ or } CH &= b, \\ CN &= x, \\ PN &= y, \end{aligned}\right\} \begin{aligned} &\text{Then, by Art. 9.} \\ &(b^2 - a^2)\times(x-a)\times(x+a) = a^2y^2, \\ &\text{or } (SC^2 - AC^2)\times AN\times NM = AC^2\times PN^2. \\ &\text{But, by construction, } SC^2 - AC^2 = BC^2; \\ &\therefore BC^2\times AN\times NM = AC^2\times PN^2, \\ &\text{and } AN\times NM : PN^2 :: AC^2 : BC^2.^* \end{aligned}$$

Therefore, as the square of the major axis is to the square of the minor, so are the rectangles of the abscissas of the former to the squares of their ordinates.

COR. 1. Since $AN\times NM = (CN - AC)\times(CN + AC) = CN^2 - AC^2$; $\therefore CN^2 - AC^2 : PN^2 :: AC^2 : BC^2$.

(82.) COR. 2. Produce NP to p, and draw any ordinate pm at right angles to Cm, then (since the conjugate hyperbola Bp is described with BC as major and AC minor axis) $Bm\times mO : pm^2 :: BC^2 : AC^2$, or (since $Bm\times mO = (Cm - BC)\times(Cm + BC)$) $Cm^2 - BC^2 : pm^2 :: BC^2 : AC^2$.

(83.) COR. 3. Since $pN = Cm$, and $pm = CN$, we have (by Cor. 2.)

$$\begin{aligned} & pN^2 - BC^2 : CN^2 :: BC^2 : AC^2, \\ \text{or } & pN^2 - BC^2 : BC^2 :: CN^2 : AC^2, \\ \text{and dividendo, } & pN^2 - 2BC^2 : BC^2 :: CN^2 - AC^2 : AC^2, \\ & :: PN^2 : BC^2. \end{aligned}$$

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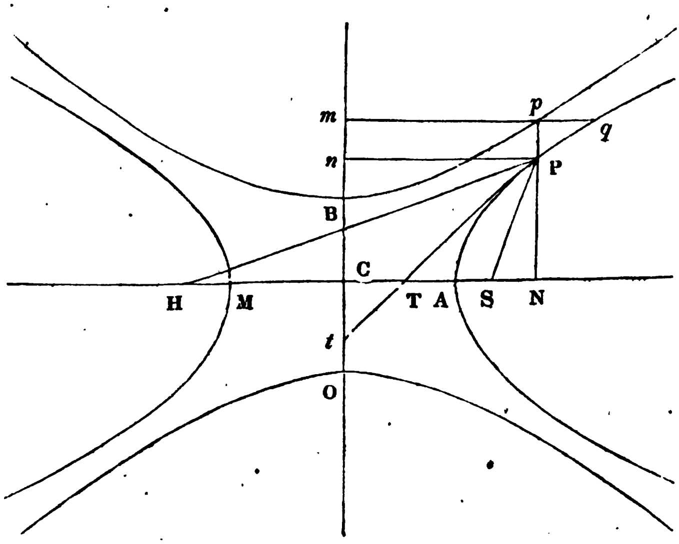
* The general property of the Hyperbola analogous to the 10th Property of the Ellipse, viz. $Pv\times vG : Qv^2 :: PC^2 : CD^2$, will be found at the end of the Properties of the Hyperbola derived from its relation to the Asymptote.

Hence $pN^2-2BC^2=PN^2$, and $pN^2-PN^2=2BC^2$; in the same manner, if mp be produced to q, it may be proved that $qm^2-pm^2=2AC^2$.

That is, the square of any ordinate to either axis is less than the square of the same ordinate produced to the conjugate Hyperbola, by twice the square of the semi-axis, to which it is parallel.

PROPERTY 7. (Prop. 8. of Ellipse.)

(84.) If the ordinate Pn be drawn to the minor axis, then $BC^2+Cn^2 : Pn^2 :: BC^2 : AC^2$.



In this case, $Pn=CN$, and $Cn=PN$; therefore $AN\times NM=(CN^2-AC^2=)Pn^2-AC^2$, and $PN^2=Cn^2$; hence, by substitution in Art. 81, Cor. 1, we have,

$$Pn^2-AC^2 : Cn^2 :: AC^2 : BC^2;$$
$$\therefore Pn^2-AC^2 : AC^2 :: Cn^2 : BC^2.$$
Componendo, $$Pn^2 : AC^2 :: BC^2+Cn^2 : BC^2,$$
or $$BC^2+Cn^2 : Pn^2 :: BC^2 : AC^2.$$

That is, as the square of the minor axis is to the square of the major, so is the sum of the squares of the semi-minor, and of the distance from the center to any ordinate upon the minor, to the square of that ordinate.

PROPERTY 8. (Prop. 11. of Ellipse.)

(85.) If the tangent PT cuts the major axis in T, and the minor axis in t, then $CN \times CT = AC^2$, and $Ct \times Cn = BC^2$. (See last Fig.)

Since PT bisects the angle HPS, by Euc. 3, 6. we have

$$HT : TS :: HP : PS;$$
$$\therefore HT-TS(2CT)^* : HT+TS(SH) :: HP-PS(2AC) : HP+PS. (A)$$

But by Euc. 12. 2, $HP^2 = HS^2 + PS^2 + 2HS \times SN$;

$$\therefore HP^2 - PS^2 = HS^2 + 2HS \times SN,$$
$$= (HS+SN)^2 - SN^2,$$
$$= HN^2 - SN^2;$$

$$\therefore HN-SN(SH) : HP-PS(2AC) :: HP+PS : HN+SN (2CN).\dagger (B)$$

Hence we have,

$$2CT : SH :: 2AC : HP+PS, (A)$$
$$SH : 2AC :: HP+PS : 2CN; (B)$$
$$\therefore 2CT : 2AC :: 2AC : 2CN,$$
and $CT : AC :: AC : CN$, or $CN \times CT = AC^2$.

(86.) Since $CN : AC :: AC : CT$ (and first : third :: first^2 : second^2) $CN : CT :: CN^2 : CA^2$;

dividendo, $NT : CT :: CN^2 - AC^2 : AC^2 :: PN^2 : BC^2$.

But by sim. $\triangle$s, PTN, TCt, $NT : CT :: PN : Ct$.

Hence, $PN : Ct :: PN^2 : BC^2$, or $PN \times Ct = BC^2$.

but $PN = Cn$, $\therefore Cn \times Ct = BC^2$.

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* For $HT-TS = HC+CT-TS = SC+CT-(SC-CT) = 2CT$.

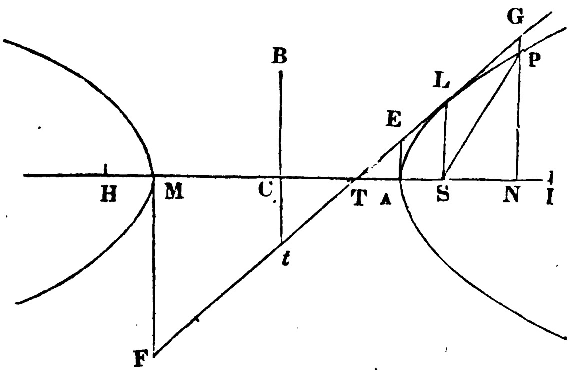
† For $HN+SN = HS+2SN = 2CS+2SN = 2(CS+SN) = 2CN$.

Therefore, if a tangent and ordinate be drawn from any point of the curve to either of the axes, half that axis will be a mean proportional between the distances of the two intersections from the center.

(87.) Cor. $CN^2-AC^2=CN^2-CN\times CT=CN\times(CN-CT)=CN\times NT$.

PROPERTY A. (Prop. C. of Ellipse.)

(87.*a*.) Let TLG be the focal tangent, or the tangent drawn at the extremity of SL, the ordinate from the focus. Let NPG be any ordinate, produced to meet the tangent TLG. Then $SP=NG$.



If AI be taken equal to SP, then $IM=HP$.

$\therefore SP=CI-CA$ and $HP=CI+CA$.

(Euc. 47. 1.) $(SP^2)(CI-CA)^2=PN^2+(CN-CS)^2(NS^2)$.

And $(HP^2)(CI+CA)^2=PN^2+(CN+CS)^2(NH^2)$.

Expand and subtract $4CI.CA=4CN.CS$, and

$CI.CA=CN.CS$. $\therefore CN : CI(SP+AC) :: CA : CS$

$:: CT : CA$ (85.)

$\therefore CN-CT(TN) : SP :: CT : CA$. (Euc. 19. 5.)

Again, $CS.CT=AC^2$, and (70.) $CS^2-AC^2(=CS.ST)=BC^2$.

$\therefore CS.CT : CS.ST$, or $CT : ST :: AC^2 : BC^2$

$:: AC : SL$ (75.)

$\therefore ST : SL :: CT : AC :: TN : SP$.

But (sim. tri.) $ST : SL :: TN : NG$;
$\therefore SP = NG$.

That is, the distance from the focus to any point of the curve, is equal to the ordinate to that point, produced until it meets the focal tangent.

Cor. 1. $AS = AE$ and $SM = MF$. Also $Ct = AC$.

For (sim. tri.) $CT : Ct :: ST : SL :: CT : AC$.

Cor. 2. Hence, also, since $TN : NG$ is a constant ratio, $TN : SP$ is a constant ratio. Therefore, if a line be drawn through T perpendicular to AC, the distance of the point P from that line ($= TN$) is in a constant ratio to SP, the distance of the same point from the focus. This ratio, being (by demonstration above) $= CT : CA$, is a ratio of less inequality. This perpendicular is the directrix of the Hyperbola. (See. Art 138. *et seq.*)

Property 9. (Prop. 12. of Ellipse.)

(88.) If PCG, DCK, be conjugate diameters of the Hyperbola, and OPF be drawn perpendicular to CD produced if necessary, then $PO \times PF = BC^2$. (See next Fig.)

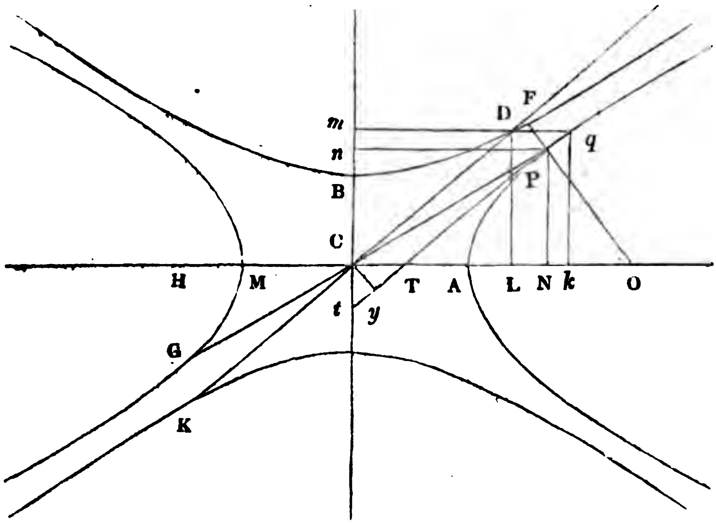
Draw Cy parallel to PF. Then because PO is parallel to Cy, and Ct parallel to PN, the right-angled triangles tCy, PON, are similar; $\therefore Ct : Cy :: PO : PN$. But $Cy = PF$, and $PN = Cn$, being opposite sides of a parallelogram;
$\therefore Ct : PF :: PO : Cn$, or $PO \times PF = Ct \times Cn = BC^2$ (86.)

Therefore, if from the extremity of any diameter, a perpendicular be drawn to its conjugate; then the rectangle of that perpendicular and the part of it intercepted by the axis major, will be equal to the square of the semi-axis minor.

Property 10. (Prop. 13. of Ellipse.)

(89.) Draw the ordinates DL, PN, to the major axis, then $CN^2 - CL^2 = AC^2$, and $DL^2 - PN^2 = BC^2$. (See next Fig.)

Draw the ordinate mD, and produce it to meet the hyperbola in q, and draw qk perpendicular to the major axis.



By Cor. 1. Art. 81. $Ck^2 - AC^2 : kq^2 :: \quad AC^2 \quad : BC^2$,

and $CN^2 - AC^2 : PN^2 :: \quad AC^2 \quad : BC^2$,

$\therefore Ck^2 - AC^2 : kq^2 :: CN^2 - CA^2 : PN^2$,

But $Ck = mq$, $mD = CL$, and $qk = DL$; and by Art. 83. $mq^2 - mD^2 = 2AC^2$; $\therefore mq^2 - AC^2 = (Ck^2 - AC^2 =) AC^2 + mD^2 = AC^2 + CL^2$; hence, by substitution, we have,

$$AC^2 + CL^2 : DL^2 :: CN^2 - AC^2 : PN^2, \quad \text{(A.)}$$

by sim. $\triangle$s, DCL, PTN,

$$DL^2 : CL^2 :: \quad PN^2 \quad : NT^2,$$

$\therefore$ ex æquo, $AC^2 + CL^2 : CL^2 :: CN^2 - AC^2 : NT^2$,

$$:: \quad CN \times NT \quad : NT^2, \quad (87.)$$

$$:: \quad CN \quad : NT.$$

Hence $AC^2 + CL^2 : AC^2 :: \quad CN \quad : CN - NT (CT)$

$$:: \quad CN^2 \quad : CT \times CN,$$

$$:: \quad CN^2 \quad : AC^2 \quad (85.)$$

$$\therefore AC^2+CL^2=CN^2,$$
$$\text{or } CN^2-CL^2=AC^2.$$

(90.) Since $AC^2+CL^2=CN^2$, by substitution in proportion (A), we have,

$$\begin{array}{rlll} & CN^2 : DL^2 & :: CN^2-AC^2 & : PN^2, \\ \text{or} & CN^2 : CN^2-AC^2 & :: DL^2 & : PN^2, \\ \text{and } div^{do}. & AC^2 : CN^2-AC^2 & :: DL^2-PN^2 & : PN^2, \\ \therefore & AC^2 : DL^2-PN^2 & :: CN^2-AC^2 & : PN^2, \\ & & :: AC^2 & : BC^2; \end{array}$$
$$\therefore DL^2-PN^2=BC^2.$$

Hence, if ordinates to either axis be drawn from the extremities of any two conjugate diameters, the difference of their squares will be equal to the square of half the other axis.

Property 11. (Prop. 14. of Ellipse.)

$$(91.)\ PC^2 \backsim CD^2=AC^2 \backsim BC^2,$$

PG and DK being conjugate diameters.

For by Arts. 89, 90. $CN^2-CL^2=AC^2$,

$$\text{and } DL^2-PN^2=BC^2,$$
$$\therefore CN^2+PN^2-(CL^2+DL^2)=AC^2-BC^2,$$
$$\text{or } CP^2-CD^2=AC^2-BC^2.$$

Hence the difference of the squares of any two semi-conjugate diameters is equal to the difference of the squares of the semi-axes.

Property 12. (Prop. 15. of Ellipse.)

$$CD\times PF=AC\times BC.$$

In Art. 90. it was proved that

$$\begin{array}{rllll} & CN^2 & : DL^2 & :: AC^2 & : BC^2; \\ \therefore & CN & : DL & :: AC & : BC. \\ \text{or} & CN & : AC & :: DL & : BC. \end{array}$$

But by sim. $\triangle$s, TCy, DCL, CT : Cy (PF)::CD : DL.

$$\begin{aligned} \text{Hence we have, CN} &: \text{AC} :: \text{DL} : \text{BC}, \\ \text{and CT} &: \text{PF} :: \text{CD} : \text{DL}; \\ \therefore \text{CN}\times\text{CT}(\text{AC}^2) &: \text{AC}\times\text{PF} :: \text{CD} : \text{BC}, \\ \text{or AC} &: \text{PF} :: \text{CD} : \text{BC}; \\ \therefore \text{CD}\times\text{PF} &= \text{AC}\times\text{BC}. \end{aligned}$$

That is, if from the extremity of any diameter a perpendicular be drawn to its conjugate, the rectangle of that perpendicular and the semi-conjugate is equal to the rectangle of the semi-axes.

(92.) Cor. Hence it appears, that all the parallelograms inscribed in the hyperbolas, and having their sides drawn through the extremities of any diameter and its conjugate, are equal to each other and to the parallelogram described about the major and minor axes; for the parallelogram *eghf* (See Fig. in page 53.) described about the conjugate diameters PCG, DCK, is equal to four times *e*DCP= 4CD$\times$PF=4AC$\times$BC=right-angled parallelogram whose sides are 2AC and 2BC=parallelogram described about the major and minor axes.

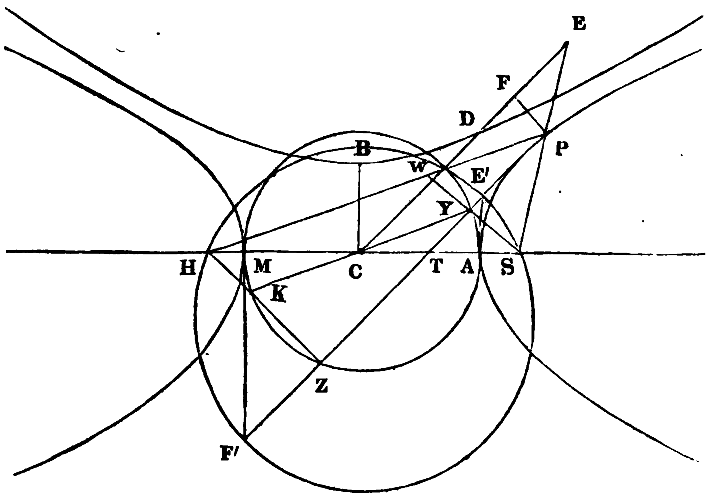
Property 13. (Prop. 16. of Ellipse.)

(93.) If SY, HZ, be perpendiculars drawn from the foci to the tangent PYZ, then the points Y and Z are in the circumference of a circle described upon the major axis AM.

Join YC, and produce SY to meet HP in W.

Since the tangent PYZ bisects the $\angle$HPS; in the right-angled triangles WPY, SPY, we shall have PW=PS, and WY=YS. Now since WY=YS, and HC=CS, CY must be parallel to HW, and $\therefore$ SC : SH::CY : HW; but SC=$\frac{1}{2}$SH; $\therefore$CY=$\frac{1}{2}$HW=$\frac{1}{2}$(HP$-$PW)=$\frac{1}{2}$(HP$-$PS)=$\frac{1}{2}$AM=AC; $\therefore$Y is a point in the circle whose center is C, and radius CA. In the same manner it might be proved that Z is a point in the same circle.

Hence, if perpendiculars be dropped from the foci upon any tangent to the hyperbola, the intersections of those perpendiculars with



the tangent will be in the circumference of a circle described upon the axis major.

PROPERTY 14. (Prop. 17. of Ellipse.)

(94.) $SY \times HZ = BC^2$.

Since the $\angle HZP$ is a right angle, it must be in a semi-circle; if $\therefore$ YC is produced to meet HZ in K, K will be in the circumference of the circle, and YK will be a diameter. Hence $YC = CK$; and as $SC = CH$, and $\angle SCY = \angle KCH$, the side HK must be equal to SY. By the property of the circle (Euc. 36. 3.) $HK \times HZ = HM \times HA = BC^2$, (74.) Hence (since $HK = SY$) $SY \times HZ = BC^2$

That is, the rectangle of the perpendiculars from the foci upon any tangent is equal to the square of the semi-axis minor.

(95.) Cor. By sim. $\triangle$'s, SPY, HPZ,

$$SP : SY :: HP : HZ = \frac{SY \times HP}{SP};$$

$$\therefore SY \times HZ = \frac{SY^2 \times HP}{SP} = BC^2.$$

$$\text{and} \quad SY^2 = BC^2 \times \frac{SP}{HP};$$

$$\therefore SY = BC \times \sqrt{\left(\frac{SP}{HP}\right)} \propto \sqrt{\left(\frac{SP}{HP}\right)}.$$

Property B. (Prop. D. of Ellipse.)

(95.*a*.) Let the vertical tangents AE′, MF′ be drawn; then E′A.F′M=BC², and E′F′ is the diameter of a circle passing through S and H.

By sim. tri. $E'A : SY :: TA : TY.$

and $HZ : F'M :: TZ : TM.$

But (Euc. 35. 3.) $TA.TM = TY.TZ.$

Or, $TA : TY :: TZ : TM.$

$\therefore E'A : SY :: HZ : F'M,$

and $E'A.F'M = SY.HZ = (94.) BC^2.$

Again (74.) $AH.HM = BC^2 = E'A.F'M,$

$\therefore AH : E'A :: F'M : HM.$

Hence (Euc. 6. 6.) the triangles E′AH and HF′M are similar, and $\angle E'HA = \angle HF'M$ and $\angle F'HM = \angle AE'H$. Whence $\angle E'HF'$* is a right angle, and a circle described on E′F′ will pass through H. The same may also be shown of S.

Wherefore, if tangents be drawn from the vertices, to meet any other tangent, the rectangle of the vertical tangents will be equal to the square of the semi-axis minor; and the intercepted part of the other tangent will be the diameter of a circle passing through the foci.

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* The points E′H, HF′ should be joined in order to form the triangles E′AH, HMF′.

PROPERTY 15. (Prop. 18. of Ellipse.)

(96.) Draw the semi-conjugate diameter CD, then $SP \times HP = CD^2$.

By sim. $\triangle$s, SPY, HPZ, PEF,

$$SP : SY :: PE : PF,$$
$$\text{and } HP : HZ :: PE : PF;$$
$$\therefore SP \times HP : SY \times HZ(BC^2) :: PE^2(AC^2) : PF^2;$$
$$\therefore SP \times HP = \frac{AC^2 \times BC^2}{PF^2}.$$

But by Property 12. $CD \times PF = AC \times BC$;

$$\therefore CD^2 = \frac{AC^2 \times BC^2}{PF^2}.$$

Hence $SP \times HP = CD^2$.

Or, the rectangle contained by the straight lines drawn from the foci to the extremity of any diameter, is equal to the square of half the conjugate to that diameter.

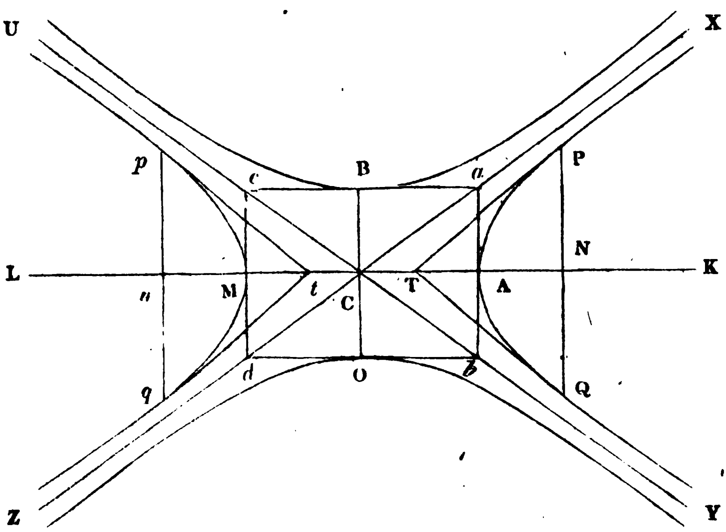
IX.

On the properties of the Hyperbola derived from its relation to the Asymptote.

The properties of the Hyperbola hitherto exhibited are perfectly analogous to those of the Ellipse; we proceed now to explain some of the properties in which these two curves essentially differ. But we must first show what is meant by an Asymptote.

(97.) Since the two branches of the opposite hyperbolas are precisely equal and similar on each side of the axis LK, if two ordinates PQ pq be drawn at equal distances AN, Mn, from the points A, M, then the tangents to the points P, Q will meet in the same point T, and tangents to the points p, q in the same point t. Now by Art. 85, $CN \times CT = AC^2$, and since AC is a constant quantity, CT varies inversely as CN; when CN therefore becomes infinite CT will be

equal to 0; i. e. if P, Q, are points in the curve at an infinite distance, the tangents PT, QT will meet in C;* for the same reason



if p, q are points, at an infinite distance in the opposite hyperbola, then the tangents pt, qt will also meet in C; and since the $\angle$PTQ$=$ $\angle ptq$, these four lines will evidently then coalesce into two, viz. PT with tq and pt with TQ. The tangents to the two opposite hyperbolas at an infinite distance, may therefore be represented by two lines XCZ, UCY, intersecting each other in C, and making equal angles XCK, UCL, KCY, ZCL with the axis. These lines XCZ, UCY are called Asymptotes; and we are now to determine their position with respect to the axes of the hyperbolas.

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* To make this more intelligible, conceive PN to move parallel to itself in the direction NK, then since CN$\times$CT$=$a constant quantity, whilst CN varies through all degrees of magnitude, the point T will only pass from T to C so as to make CT$=$0; i. e. when P is a point in the curve at an infinite distance, the tangent PT will pass through C; and so of the rest.

(98.) Draw Aa at right angles to AM. When P is removed to an infinite distance, the triangle PNT becomes similar to the triangle aAC, and CN becomes the same as NT. Hence, in this case, PN : NT or NC :: Aa : AC (A); but by Cor. 1, Art. 81, $CN^2 - AC^2 : PN^2 :: AC^2 : BC^2$; and when CN is infinite, AC vanishes with respect to CN,* therefore this latter proportion becomes $CN^2 : PN^2 :: AC^2 : BC^2$, or CN : PN :: AC : BC (B); compare the two proportions (A) and (B), and we have Aa : AC :: BC : AC, or Aa = BC. Draw therefore Aa at right angles to AM, and make it equal to BC, join Ca, and this gives the position of the asymptote XCZ. In the same manner, by making Ab = BC, and joining Cb, we determine the position of the asymptote UCY; indeed, from what has been proved, it appears, that if a parallelogram *acdb* be described about the major and minor axes, the asymptotes will be merely the prolongation of the diagonals of such parallelogram.

(99.) These lines XCZ, UCY, will also be asymptotes to the conjugate hyperbolas; for by a similar process of reasoning it might be shown, that the position of their asymptotes would be determined by drawing perpendiculars Ba, Ob, at B and O, and making Ba and Ob each equal to AC. Thus these four hyperbolas are inclosed as it were between their asymptotes; and by producing the ordinates to meet these asymptotes, new properties of the curves will arise, which we shall now proceed to investigate.

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* To show that in this case $CN^2 - AC^2$ may be considered as equal to CN^2, let $CA = a$, $AN = x$, then $CN = x + a$, and $CN^2 = x^2 + 2ax + a^2$; hence $CN^2 - AC^2 (= CN^2 - a^2) = x^2 + 2ax$; we have therefore $CN^2 : CN^2 - AC^2 :: x^2 + 2ax + a^2 : x^2 + 2ax :: x + 2a + \frac{a^2}{x} : x + 2a$; but when x is infinite, $\frac{a^2}{x}$ becomes equal to 0; in this case, therefore, this latter ratio becomes a ratio of equality, from which it follows that CN^2 may be substituted for $CN^2 - AC^2$.

Property 16.

(100.) Let the ordinate Pp be produced to meet the asymptotes in the points L, l; then $PL.Pl=BC^2$ and $pl.pL=BC^2$; also $PL.Lp=BC^2$ and $pl.lP=BC^2$.

By Cor. 1. Art. 81,

$$CN^2-CA^2 : PN^2 :: AC^2 : BC^2.$$

By sim. $\triangle$s, LNC, aAC,

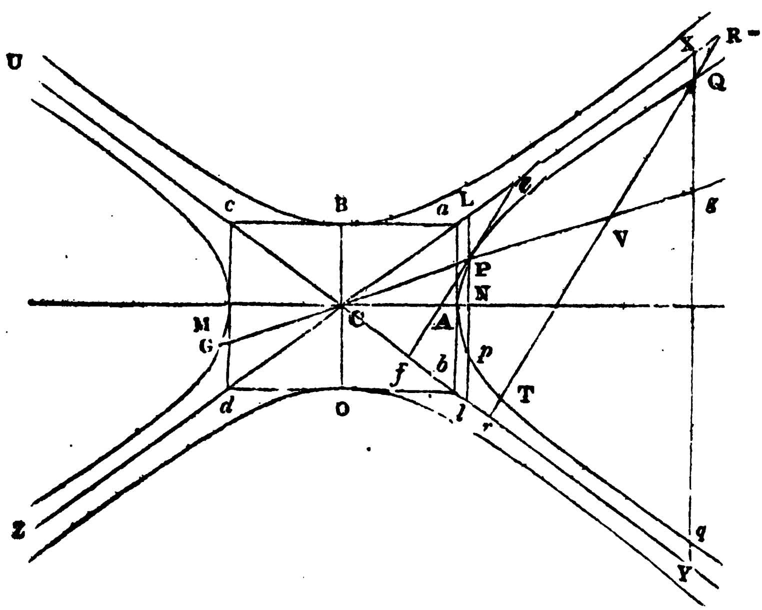
$$CN^2 : LN^2 :: AC^2 : Aa^2 \ (BC^2)$$

$$\therefore CN^2 : CN^2-CA^2 :: LN^2 : PN^2,$$

and dividendo, $AC^2 : CN^2-CA^2 :: LN^2-PN^2 : PN^2$;

or $AC^2 : LN^2-PN^2 :: CN^2-CA^2 : PN^2,$

$$:: AC^2 \cdot BC^2;$$



$$\therefore LN^2-PN^2=BC^2.$$

But $LN^2-PN^2=(LN-PN)\times(LN+PN)=PL\times Pl$;

$$\therefore PL\times Pl=BC^2 \text{ or } PL.Lp=BC^2.$$

For the same reason, $pl\times pL=BC^2$ or $pl.lP=BC^2$.

Therefore, if an ordinate to the axis major be produced to meet the asymptotes, then the rectangle of the segments intercepted between the curve and either asymptote will be equal to the square of the semi-axis minor.

(101.) Cor. 1. Hence $PL \times Pl = pl \times pL = PL.Lp = pl.lP$.

Cor. 2. Draw any other ordinate Qq, and produce it to meet the asymptotes in X and Y, then will $QX \times QY = Aa^2$; hence we have $QX \times QY = PL \times Pl$.

Property 17.

(102.) Draw any diameter PCG, and produce it to g; draw the ordinate QT to that diameter, and produce it to meet the asymptotes R, r; then $QR \times Qr = Tr \times TR$. (See last Fig.)

Through the points P, Q, draw Ll, XY perpendicular to the axis of the hyperbola, and draw the tangent ef at P.

By sim. triangles QXR, PLe; QrY, Pfl; we have

$$QX : QR :: PL : Pe,$$
$$\text{and } QY : Qr :: Pl : Pf;$$
$$\therefore QX \times QY : QR \times Qr :: PL \times Pl : Pe \times Pf.$$

But by Art. 101, $QX \times QY = PL \times Pl$; hence $QR \times Qr = Pe \times Pf$.

In the same manner, by drawing an ordinate through T perpendicular to the axis, it might be shown that $Tr \times TR = Pe \times Pf$; hence $QR \times Qr = Tr \times TR$.

Therefore, if an ordinate to any diameter be produced to meet the asymptotes, the rectangle of the segments intercepted between the curve and one asymptote, will be equal to the rectangle of the segments intercepted between the curve and the other.

(103.) Cor. 1. Since $Qr = QT + Tr$, and $TR = QT + QR$, we have

$$QR \times (QT + Tr) \quad = Tr \times (QT + QR),$$
$$\text{or } QR \times QT + QR \times Tr = Tr \times QT + Tr \times QR.$$

Subtract $QR \times Tr$ from each side of this latter equation, and there results $QR \times QT = Tr \times QT$, from which it appears that $QR = Tr$;

in the same manner it may be proved that $QX=qY$, and $PL=pl$. If therefore Rr moves parallel to itself till it comes into the position of the tangent ef (in which case the points Q and T coincide in P), we shall have $Pe=Pf$, and consequently $Pe\times Pf=Pe^2$.

(104.) Cor. 2. Since the triangles eCf, RCr are similar, and since the diameter GCg bisects ef in P, it will bisect Rr in V; hence $VR=Vr$; and as $QR=Tr$, we have $VQ=VT$, i. e. the diameter GCg bisects all its ordinates.

(105.) Cor. 3. Hence $VR^2-VQ^2=Pe^2$. For $VR^2-VQ^2=(VR-VQ)\times(VR+VQ)=(VR-VQ)\times(VR+VT)=QR\times RT=QR\times Qr$, (for $RT=QR+QT=Tr+QT=Qr$). But $QR\times Qr=Pe\times Pf=(103)Pe^2$; $\therefore VR^2-VQ^2=Pe^2$.

Property 18.

(106.) Join AB, and let it cut the asymptote XCZ in S; draw PD parallel to the asymptote UCY, cutting the asymptote XCZ in R; then $CR\times RP=AS^2$. (Fig. in next page.)

Since the diagonals BA, aC of the parallelogram aBCA are equal and bisect each other in the point S, the lines SA, SC, SB, Sa are equal; hence the $\angle SAC=\angle SCA$; but $\angle SCA=\angle ACY$, $\therefore$ $\angle SAC=\angle ACY$, and consequently AB is parallel to UCY. If therefore Pr, Am are drawn parallel to the asymptote XCZ, then $PRCr$, $ASCm$ will be parallelograms, and Pr will be equal to CR, and Am to SC.

By sim. triangles Prl, Amb; PRL, ASa; we have

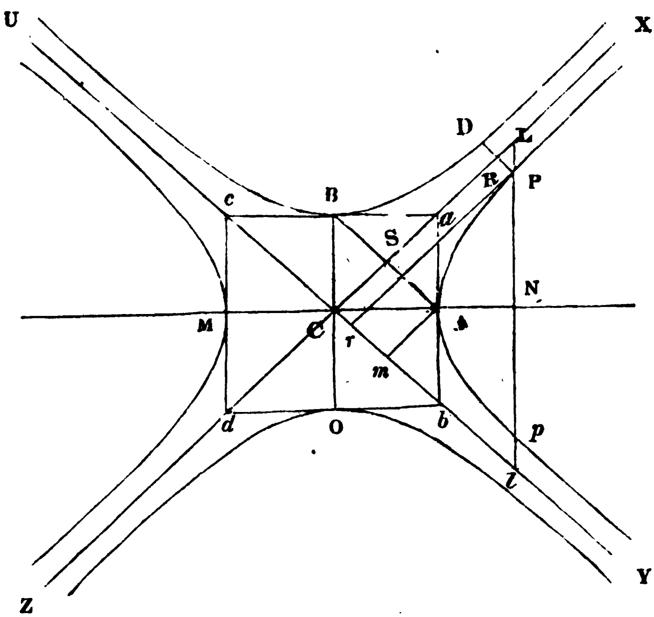
$$Pr(CR) : Pl :: Am(SC) : Ab,$$
$$\text{and } RP : PL :: SA : Aa;$$
$$\therefore \overline{CR\times RP} : \overline{PL\times Pl} :: \overline{SC\times SA} : \overline{Aa\times Ab}.$$

But $PL\times Pl=Aa\times Ab$; hence $CR\times RP=SC\times SA=SA^2$.

That is, if from any point of the curve a line be drawn to the nearer asymptote, parallel to the other asymptote, the rectangle of this line, and the distance of its intersection with the asymptote from the center, is a constant quantity; and is equal to the square of half the diagonal of the rectangle of the semi-axes.

(107.) Cor. 1. Since XCZ is likewise an asymptote to the conjugate hyperbola, by a similar process of reasoning it might be shown that $CR \times RD = SB^2 = SA^2$; hence $CR \times RD = CR \times RP$, and consequently $RD = RP$, i. e. PD is bisected by the asymptote.

(108.) Cor. 2. Since $SA = \frac{1}{2}AB$, SA^2 is a constant quantity; hence RP varies inversely as CR; when CR therefore is infinite, RP will become equal to 0; which coincides with what has already been said as to the asymptotes touching the curve at an infinite distance.

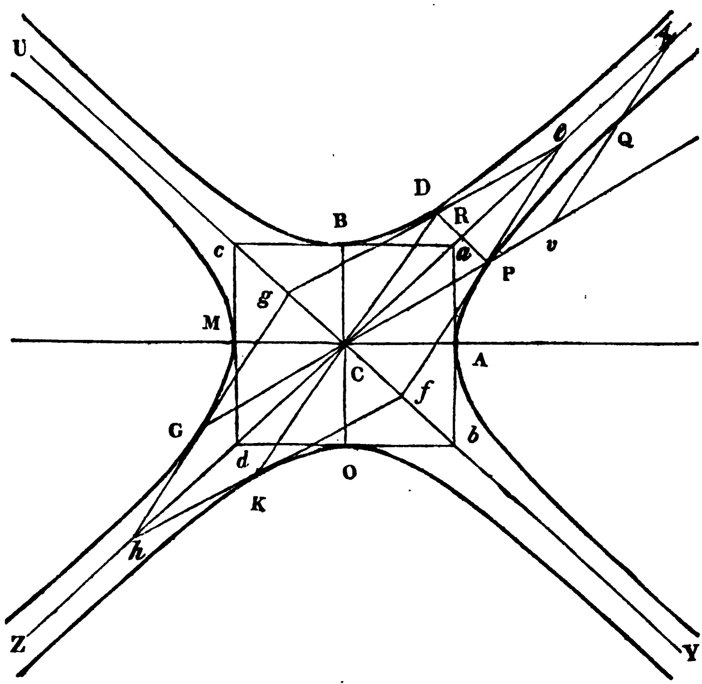


Property 19.

Join CD, and produce it to K; draw the diameter PCG; then will DCK be the conjugate diameter to PCG. (Fig. in next page.)

(109.) Draw ef touching the curve in the point P, and meeting the asymptotes XCZ, UCY in the points e and f; then by Art. 103, Pf will be equal to Pe; and since PR is parallel to Cf, CR will be

also equal to R*e*. Hence, in the triangles CRD, PR*e*, we have CR=R*e*, RD=RP, and ∠ CRD= ∠ *e*RP, ∴ (Euc. 4. 1.) CD is equal to P*e*, and the ∠ DCR equal to the ∠ R*e*P ; consequently DCK is parallel to the tangent *ef*, and is therefore the conjugate diameter to PCG (73.)



Therefore, if a parallel to either asymptote cut conjugate Hyperbolas, the diameters passing through the points of intersection will be conjugate to each other.

(110.) Cor. Join D*e*, then *e*DCP will be a parallelogram,* whose diagonal is C*e*; and as D*e* is parallel to the diameter PCG, it touches the conjugate hyperbola in D. Complete the parallelogram *eghf*,

* For CD being equal and parallel to P*e*, D*e* must be equal and parallel to CP. (Euc. 33. 1.)

as in Art. 79, then in the same manner as it has been proved that Ce is the diagonal of the parallelogram $eDCP$, it might also be proved that the point h would be found in the asymptote XCZ, and the points g, f in the asymptote UCY; these asymptotes are therefore the prolongation not only of the diagonals of the parallelogram described about the major and minor axes, but also of the parallelogram described about any two conjugate diameters.

PROPERTY 20. (Prop. 10. of Ellipse.)

(111.) Draw the ordinate Qv, then $Pv \times vG : Qv^2 :: PC^2 : CD^2 :: PG^2 : DK^2$.

Produce vQ to X, then, by sim. $\triangle$s, CvX, CPe.

$$Cv^2 : CP^2 :: vX^2 : Pe^2;$$
$$\therefore Cv^2 - CP^2 : CP^2 :: vX^2 - Pe^2 : Pe^2,$$
$$\text{and } Cv^2 - CP^2 : vX^2 - Pe^2 :: CP^2 : Pe^2. \text{ (A)}$$
$$\text{But } Cv^2 - CP^2 = (Cv - CP) \times (Cv + CP) = Pv \times vG.$$
$$\text{By Art. 105. } vX^2 - vQ^2 = Pe^2, \therefore vX^2 - Pe^2 = vQ^2.$$
$$\text{Now (109.) } Pe^2 = CD^2.$$

Hence, by substitution in Proportion (A), we have

$$Pv \times vG : Qv^2 :: CP^2 : CD^2 :: PG^2 : DK^2.$$

Therefore, the square of any diameter is to the square of its conjugate, as the rectangles of its abscissas are to the squares of their ordinates.

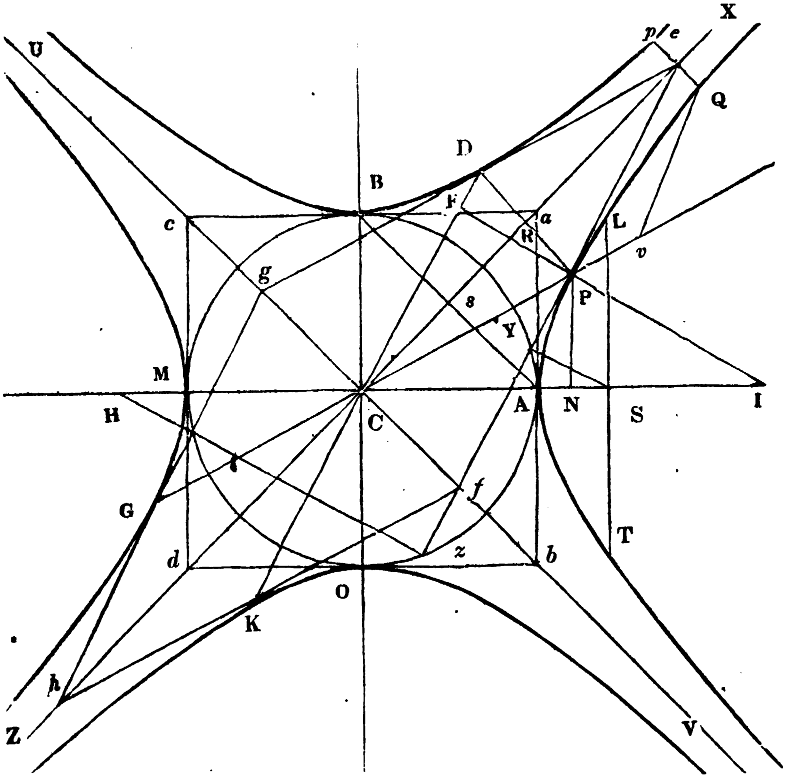
By reasoning similar to that employed in Arts. 54.*a*, 67.*a*., and 67.*b*., properties may be inferred, analogous to Props. B, E and F of the Ellipse.

X.

On the Properties of the EQUILATERAL *Hyperbola.*

In Art. 72, it was observed, that if the axes of the hyperbola become equal, it is then said to be equilateral; in this case the figure possesses some peculiar properties, which it may be worth while to investigate.

(112.) Let the annexed figure represent an equilaterial hyperbola, with its opposite and conjugate hyperbolas; then since the axes ACM, BCO are equal, it is evident that if a circle be



described upon the axis ACM, it will pass through the extremities of the axis BCO, and that the rectangular figure *abdc* which circumscribes those axes will be a square. Draw the diagonals *ad*, *cb*, and produce them each way to X, V, U, Z; then XCZ, UCV will be the asymptotes to the four hyperbolas; and as the angles *a*CB, *c*CB, are each of them half a right angle, the angle *a*C*c* will be a right angle. Since the asymptote XCV cuts the asymptote UCV at right angles in the center C, it will also cut all other lines BA, DP, *p*Q &c. (drawn parallel to UCV) at right angles. Now by Art. 106, $CR \times RP = sA^3$; and for the same reason $Ce \times eQ = sA^2$; $\therefore CR \times$

$RP = Ce \times eQ$, or $CR : Ce :: eQ : RP$; hence if any points R, e, &c. are taken in the asymptote, and from them ordinates PR, eQ, &c. are drawn at right angles to it, then the abscissas CR, Ce, &c. will be to each other inversely as the ordinates RP, eQ, &c.*

(113.) Since the Latus-rectum is a third proportional to the major and minor axes; when those axes are equal, it must be equal to either of them; LST is therefore equal to ACM or BCO. Now $MS \times SA = BC^2 =$ (since $BC = AC$) AC^2; hence AC is a mean proportional between MS and SA; and since $SY \times Hz = BC^2 = AC^2$, AC is a mean proportional between the perpendiculars SY and Hz.

(114.) By Art. 91. $PC^2 - CD^2 = AC^2 - BC^2$; but $AC^2 - BC^2 = 0$, $\therefore PC^2 - CD^2 = 0$, consequently $PC = CD$, and the diameter $PCG =$ conjugate DCK. The sides eg, gh, hf, fe of the parallelogram $eghf$, drawn about those diameters, will therefore be equal; and the parallelogram itself, a Rhombus whose area will be equal to the area of the square $abdc$ described about the axes.

(115.) Draw PI at right angles to a tangent at P, and produce it to F; then by Art. 88, $PI \times PF = BC^2 = AC^2$; but $CD \times PF = AC \times BC = AC^2$; $\therefore PI \times PF = CD \times PF$, and $PI = CD = PC$, i. e. the normal PI is equal to the distance PC from the center.

---

* Let Cs or $sA = a$, $CR = x$, $RP = y$, then (since $CR \times CP = sA^2$), $xy = a^2$, and $y = \frac{a^2}{x}$, $\therefore y\dot{x} = \frac{a^2\dot{x}}{x}$, whose fluent is $a^2 \times \log. x +$ Cor.; suppose therefore the area $AsRP$ to begin from s, it would vanish when $x = Cs$ or a; hence $a^2 \times \log. a +$ Cor. $= 0$, and Cor. $= -a^2 \times \log. a$, the area $AsRP$ is therefore equal to $a^2 \times \log. x - a^2 \times \log. a = a^2 \times \log. \frac{x}{a}$. Suppose now that $Cs = a = 1$, then a^2 and a would each be equal to 1, and we should have area $AsRP = \log. x$; and thus if the abscissas CR, Ce, &c. are taken equal to the natural numbers in succession, the corresponding areas $AsRP$, $AseQ$, &c. will be the Logarithms of those numbers. It is from this circumstance that the system of logarithms whose modulus is unity are called Hyperbolic Logarithms.

(116.) Since $AN \times NM : PN^2 :: AC^2 : BC^2$, and $AC^2 = BC^2$, $\therefore AN \times NM = PN^2$. Also, by Art. 111, $Pv \times vG : Qv^2 :: PC^2 : CD^2$; but $PC^2 = CD^2$, $\therefore Pv \times vG = Qv^2$. Hence the rectangle of the abscissas is equal to the square of the ordinate, whether the ordinates be referred to the axis or to any diameter; in this respect, therefore, the properties of the equilateral hyperbola are analogous to those of the circle.

We have just hinted at the analogy which obtains between the properties of the circle and those of the equilateral hyperbola when considered in a geometrical point of view; but it appears more striking when the nature of those curves is expressed algebraically.* To pursue the inquiry respecting this analogy, would lead to investigations, which, though extremely curious and interesting in themselves, are quite foreign to our present purposes. We therefore now proceed to consider the nature of the Curvature of the three Conic Sections.

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* Let $CA = a$, $CN = x$, $PN = y$; then, by Art. 81. Cor. 1. (since $CA^2 = BC^2$ and $\therefore CN^2 - CA^2 = PN^2$) we have $y^2 = x^2 - a^2$, or $y = \sqrt{(x^2 - a^2)}$; now let a be the radius of a circle, x the abscissa measured from the center and y the ordinate, then, by the property of the circle, $y = \sqrt{(a^2 - x^2)} = \sqrt{(-1)} \times \sqrt{(x^2 - a^2)}$; the algebraic expression therefore for the ordinate of the circle is the same with the expression for the ordinate of the equilateral hyperbola, except as to the imaginary factor $\sqrt{(-1)}$. This similarity in the algebraic expressions for the ordinate, lays the foundation of some very curious analytical Theorems with respect to the analogy between these two curves.

CHAPTER V.

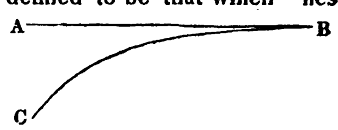
ON THE CURVATURE OF THE CONIC SECTIONS.

In order to become thoroughly acquainted with the geometry of curvilinear figures, it is necessary to acquire clear and distinct ideas of the nature of Curvature. Previous to the investigation, therefore, of the theorems relating to the curvature of the Conic Sections, it will be very proper to consider the nature of Curvature in General.

XI.

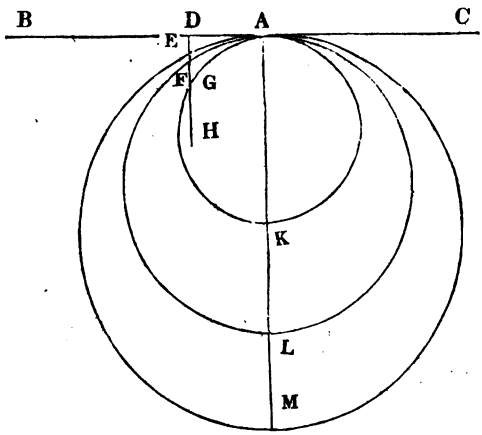
On Curvature, and the Variation of Curvature.

(117.) As a straight line (AB) is defined to be that which "lies evenly between its extreme points,"* so a curved line (BC) may be said to be that which does not "lie evenly between those points;" and by curvature is meant the continued deviation from that evenness of po-



* This is the original definition of Euclid, and it is retained by Simson, in his edition of that Geometer's works. If, however, we were left to conceive of a straight line solely from this definition, it is questionable whether our conceptions would be very clear. "The word *evenly*," as Playfair remarks, "stands as much in need of an explanation, as the word *straight*, which it is intended to define." The definition given by this latter mathematician is this. "If two lines are such, that they cannot coincide in any two points without coinciding altogether, each of them is called a straight line." A

sition which takes place in the course of its description. The curvature, moreover, is said to be greater or less, according as that deviation is greater or less within a given distance of the point from which the curve begins to be described. We know not how to illustrate this definition better, than by referring the reader to the annexed figure, where several circles AEM, AFL, AGK, &c. of different diameters AM, AL, AK, &c. begin to be described from the point A, all touching the straight line BC. At the given distance AD from the point A, draw the line DH at right angles to AD, and cutting the circles in the points E, F, G, &c., then the deviations of the cir-



cles AGK, AFL, AEM, &c. from the right line AB, are measured by the lines DG, DF, DE, &c. respectively; and since DG is greater than DF, DF than DE, &c. the curvature of the circle

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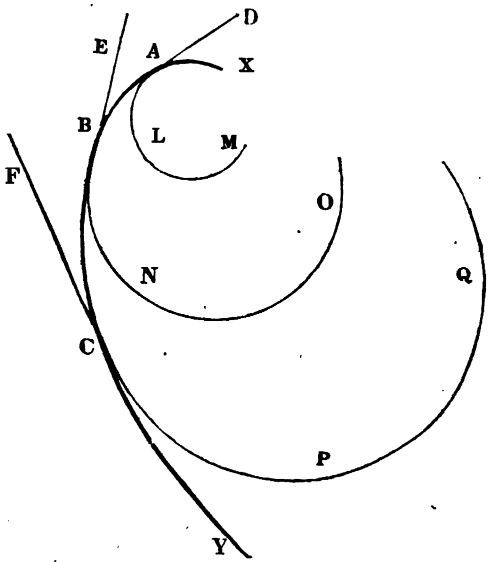
straight line being thus defined, the best account that can be given of a curve is to say, that *it is a line, which cannot have a common segment with a straight line;* or a line which continually *deviates from a straight line.*

AGK is said to be greater than that of the circle AFL, of AFL greater than that of AEM, &c. &c.*

(118.) Suppose now XABCY to be any curve, to which tangents DA, EB, FC, &c. are drawn at the points A, B, C, &c.; then, from what has been shown in Art. 117, it is evident that an unlimited number of circles may be described at each of the points A, B,



C, &c. to which the lines DA, EB, FC, &c. shall be tangents as well as to the curve; but that there can be only one circle, which shall have the same deviation from the tangent as the curve at each point. Let ALM, BNO, CPQ, &c. be the circles which have the same deviation (i. e. which coincide) with the curve at the points A,

* We have here to observe, that although the lines DE, DF, DG, &c. are made use of to illustrrte what is meant by greater or lesser curvature, yet the actual relation between the curvatures of these circles can only be ascertained by finding the relation of DE, DF, DG, &c. just at the point of contact.

B, C, &c.; then these circles are called the Circles of Curvature to those points.*

(119.) The change which takes place in the curvature from the circumstance of its being measured at different points A, B, C, &c. by circles of different diameters, is called the variation of curvature of the curve ABC.

Having thus defined what is meant by curvature and the variation of curvature, we are next to investigate the relation which takes place between the curve and the tangent just at the point of contact. This is a subject of considerable difficulty, inasmuch as it involves the consideration of quantities which will not admit of strict geometrical comparison, but require a species of minute analysis, the principles of which are exhibited in the following Theorems.

Theorem 1.

(120.) In the circle PQVL, take any arc QP; from P, Q, draw any chords PV, QV, and the tangent PR to the point P; from Q draw QR parallel to PV, and Qv parallel to RP; join QP; then, at the point of contact, the arc QP, the chord QP, the tangent RP, and the ordinate Qv, all become equal to each other.

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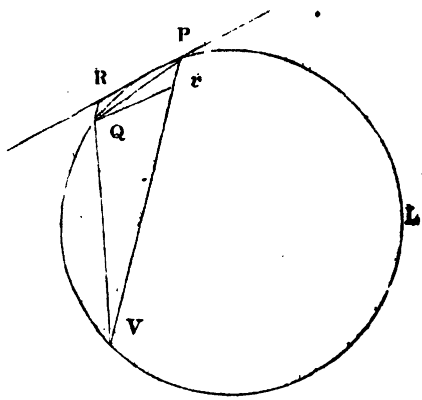
* Since the curve and circle of curvature have the same deviation from the tangent, at the point of contact, it is obvious that no other circle can be drawn between. This relation between the curve and circle of curvature is similar to that which exists between a circle and its tangent. Hence the circle of curvature is said to *touch* the curve. It will be observed, however that the circle often *cuts* the curve, which it is said to touch in the point of contact. This must always be the case, except at points of maximum or minimum curvature, when the circle falls wholly within or wholly without the curve.

Since RP touches the circle, and PQ cuts it, the angle RPQ is equal to the angle QVP in the alternate segment; and since QR is parallel to PV, the $\angle$ RQP= alternate $\angle$ QPV; the triangles PQR, PQV therefore are similar; hence we have PQ : PR:: PV : QV. Now suppose the chord PV to remain fixed whilst the chord QV revolves round the point V by the continual approach of the point Q towards P, then it is evident that the chords PV and QV continually approach towards a state of equality; PQ and PR therefore, which are to each other in the ratio of PV : QV, must also approach to a state of equality; as must also the arc QP which lies between PQ and PR, and the ordinate Qv which is equal to PR. At the point of contact, QV becomes actually equal to PV; hence the arc QP, chord QP, tangent PR, and ordinate Qv, (whose relation is expressed by the equality of the determinate lines PV, QV) must at that point become actually equal.



Theorem 2.

(121.) The chord PV is equal to $\frac{(\text{arc QP})^2}{\text{QR}}$; assuming the relation which QP and QR have to each other at the point of contact.

By similar triangles, QPR, PQV, QR : PQ::PQ : PV; but (by Art. 120.) at the point of contact, chord PQ=arc PQ, $\therefore$ in this case QR : arc PQ::arc PQ : PV; hence PV$=\frac{(\text{arc QP})^2}{\text{QR}}$.

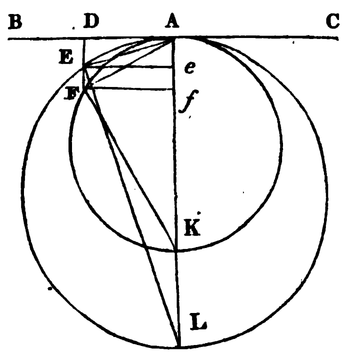
In order to remove the objection which may arise from the circumstance of representing the definite quantity PV by the quantity

$\frac{QP^2}{QR}$, in which QP and QR are confessedly too small for geometrical comparison, it should be recollected that the measure of a ratio is entirely independent of the terms of a ratio, and consequently that the two ratios which compose the proportion QR : PQ::PQ : PV are as much real ratios at that particular period when the arc PQ may be considered as equal to the chord PQ, as at any other period of the progress of the point Q towards P. The conclusion therefore deduced from the reality of that proportion, viz. that PV is equal to $\frac{PQ^2}{QR}$, must be true in the case when the arc PQ=the chord PQ, i. e. at the point of contact.

Theorem 3.

(122.) In different circles the curvature varies inversely as the radii of these circles.

Let AEL, AFK be two circles having a common tangent (BC) at A: in AB take any point D, and draw DF at right angles to AB; draw the chords AE, EL; AF, FK; and let fall E*e*, F*f*, perpen-

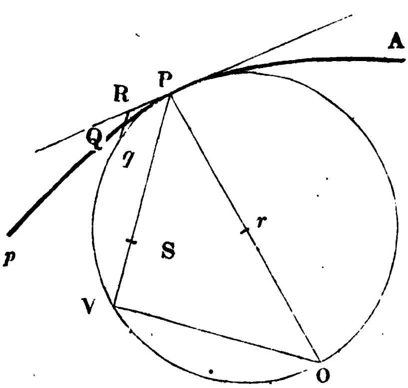


dicular to the diameter AL; then will A*e* be equal to DE, and A*f* will be equal to DF. Now (by Euc. 8. 6.) in the right-angled triangles AEL, AFK, we have A*e* : AE::AE : AL; $\therefore$ A*e* or DE$=\frac{AE^2}{AL}$; also A*f* : AF::AF : AK; $\therefore$ A*f* or DF=

$\frac{AF^2}{AK}$; hence DE : DF $::\frac{AE^2}{AL}:\frac{AF^2}{AK}$. But the curvature of the circles AEL, AFK, (see note page 80.) is measured by the relation which obtains between DE and DF just at the point of contact; and at that point, AE and AF both become equal to AD (by Art. 120.) and consequently equal to each other. At the point of contact, therefore, (since $AE^2=AF^2$) we have DE : DF $::\frac{1}{AL}:\frac{1}{AK}::$ AK : AL; i. e. curvature of circle AEL : curvature of circle AFK :: diameter of AFK : diameter of AEL :: radius of AFK : radius of AEL; i. e. the curvature in different circles varies inversely as their radii.

Theorem 4.

(123.) Let now APQ be ANY CURVE, PVO the CIRCLE OF CURVATURE to the point P; take any arc PQ and through Q draw RQq parallel to the chord PV passing through some given point S; then (assuming the relation of the quantities PQ and QR at the point of contact) PV will be equal to $\frac{PQ^2}{QR}$.



By Theorem 2, PV is equal to $\frac{Pq^2}{qR}$; but since the curve and circle of curvature coincide at the point of contact, at that point

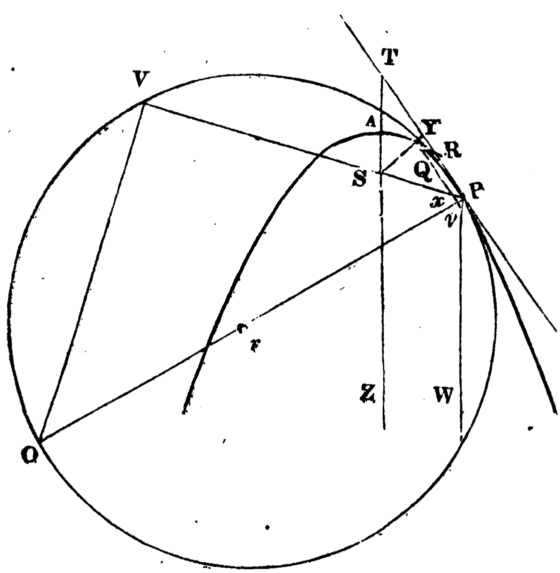
Pq will become equal to PQ, and qR equal to QR, and consequently $PV=\frac{PQ^2}{QR}$.

(124.) Draw now VO at right angles to PV, and join PO; then (PVO being a right angle, consequently in a semi-circle) PO will be the diameter of curvature to the point P. Bisect PO in r, then Pr, will be the radius and r the center of curvature to the point P.

XII.

On the curvature of the PARABOLA.

Let AQP be a Parabola, whose axis is AZ, and focus S; and let PVO be the circle of a curvature to any point P. Join SP, and produce it to meet the circle of curvature in V, then PV is the chord of curvature passing through the focus.



(125.) The Chord PV=4SP. Take any arc QP, so small that it may be considered as coinciding with the circle of curvature, and

draw QR parallel to SP; draw also Qv parallel to the tangent PT, cutting SP in x, and the diameter PW in v; then $QRPx$ will be a parallelogram, and Px will be equal to QR. Now since xv is parallel to PT, and Pv parallel to TS, the $\triangle Pxv$ is similar to the $\triangle PST$; but by Art. 17, SP is equal to ST; $\therefore Px=Pv$; hence Pv is equal to QR. Let Qv move up towards P parallel to itself, then, at the point of contact, Qv will become equal to QP;* since therefore $Pv=$ QR, and $Qv=QP$, we have $PV=\left(\frac{QP^2}{QR}\text{ by Art. 123.}=\right)\frac{Qv^2}{Pv}$. But by Art. 22, $4SP\times Pv=Qv^2$; $\therefore \frac{Qv^2}{Pv}=4SP$; hence $PV=4SP=$ parameter to the point P.

That is, the chord of curvature, passing through the focus, is equal to the parameter of the diameter at the point of contact.

(126.) $SA.PO^2=16SP^3$. Draw VO at right angles to PV and join PO. Then (124.) PO is the diameter of curvature, and therefore parallel to SY, which is perpendicular to the tangent PT. Hence the triangles PVO, SYP are similar.

$$\therefore PO : PV(=4SP) :: SP : SY,$$
$$PO^2 : 16SP^2 :: SP^2 : SY^2(=SA.SP, \text{ by Art. 32.})$$
$$PO^2 : 16SP^2 :: SP : SA,$$
$$\therefore SA.PO^2=16SP^3.$$

Therefore, a parallelopiped, whose base is the square of the diameter of curvature, and whose height is the distance from the focus to the vertex, is equal to 16 times the cube of the distance from the focus to the point of contact.

Cor. 1. The diameter of curvature$=\frac{4SP^{\frac{3}{2}}}{\sqrt{(SA)}}$.

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* By Art. 120, Qx becomes equal to QP; but at the point of contact P, the points x and v coincide; therefore at that point the three lines QP, Qx, Qv become equal to each other.

Cor. 2. $SA.PO^2 = \frac{PV^3}{4}$ and $PO = \frac{PV^{\frac{3}{2}}}{2\sqrt{(SA)}}$.

Cor. 3. The diameter PO (and of course the radius Pr) $\propto SP^{\frac{3}{2}}$ or $PV^{\frac{3}{2}}$, because SA is constant.

(127.) At the vertex A, where SP becomes perpendicular to the tangent, the chord and diameter of curvature will of course coincide; and in this case each of them becomes equal to 4SA, i. e. to the latus-rectum.* The diameter (and consequently the radius) of curvature is therefore the least at A; hence, by Art. 122, the curvature itself will be greatest at A; and since it varies as $\frac{1}{Pr}$, i. e. $\frac{1}{SP^{\frac{3}{2}}}$, it will keep continually decreasing as the point P recedes from A.

XIII.

On the Curvature of the Ellipse. (Fig. in next page.)

Let APM be an ellipse, PVLO the circle of curvature to the point P; join PS, PC, and produce them to meet the circle of curvature in the points V, L; draw VO, LO at right angles to PV, PL, and join PO; then PV is the chord of curvature passing through the focus; PL the chord passing through the center; and PO the diameter of curvature. Draw the conjugate diameter DCK; then,

(128.) The chord of curvature (PL) passing through the center is equal to $\frac{2CD^2}{PC}$. Take any small arc QP as before; draw QR parallel to PC, and Qv parallel to RP; then will Pv be equal to RQ. Suppose Qv to move up towards P, then, at the point of con-

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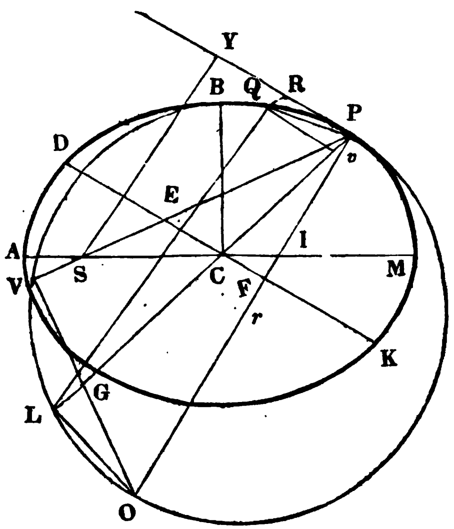
* For at A, SP becomes equal to SA; ∴ PV=4SP=4SA; and $PO = \frac{4SP^{\frac{3}{2}}}{\sqrt{(SA)}} = \frac{4SA^{\frac{3}{2}}}{\sqrt{SA)}} = 4SA.$

tact, Qv becomes equal to QP (120.), and vG becomes equal to PG, i. e. to 2PC. Now by Art. 53, $Pv \times vG : Qv^2 :: PC^2 : CD^2$; substituting therefore for Pv, vG and Qv, their values at the point of contact, we have $QR \times 2PC : QP^2 :: PC^2 : CD^2$, or $2PC : \frac{QP^2}{QR} :: PC^2 : CD^2$, $\therefore \frac{QP^2}{QR} = \frac{2PC \times CD^2}{PC^2} = \frac{2CD^2}{PC}$; but $PL = \frac{QP^2}{QR}$ (by Art. 123.), $\therefore PL = \frac{2CD^2}{PC}$.

(129.) The diameter of curvature $(PO) = \frac{2CD^2}{PF}$. The triangles PCF, PLO have a common $\angle$ at P, and right $\angle$'s at L and



F, they are therefore similar. Hence $PO : PL = \left(\frac{2CD^2}{PC}\right) :: PC : PF$, $\therefore PO = \frac{2CD^2 \times PC}{PC \times PF} = \frac{2CD^2}{PF}$; the radius of curvature (Pr) will consequently be equal to $\frac{CD^2}{PF}$.

(129.*a*.) The chord of curvature (PV) passing through the focus $=\frac{2CD^2}{AC}$. The triangles PEF, PVO have a common $\angle$ at P and right $\angle$ s at V and F, they are therefore similar. Hence PV : PO $\left(\frac{2CD^2}{PF}\right)$ :: PF : PE (AC), $\therefore PV=\frac{2CD^2\times PF}{PF\times AC}=\frac{2CD^2}{AC}$. At B (the extremity of the minor axis) the semi-conjugate becomes equal to AC; hence the chord of curvature passing from B through the focus $S=\frac{2AC^2}{AC}=2AC$.

(130.) At A (the extremity of the major axis) the diameter of curvature $\left(\frac{2CD^2}{PF}\right)=\frac{2BC^2}{AC}=$ latus-rectum;* at B (the extremity of the minor axis) it is equal to $\frac{2AC^2}{BC}$; it is therefore least at A, and greatest at B; hence, by Art. 122, the curvature is greatest at the extremity of the major axis, and least at the extremity of the minor axis. At the intermediate points between the extremities of the axes, the curvature varies inversely as the cube of the normal.†

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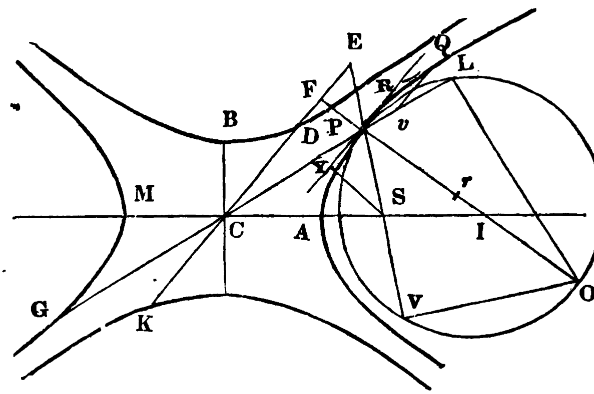
* For by Art. 39, major axis : minor axis :: minor axis : latus-rectum, i. e. 2AC : 2BC :: 2BC : latus-rectum $=\frac{4BC^2}{2AC}=\frac{2BC^2}{AC}$.

† The radius of curvature $=\frac{CD^2}{PF}$. Now by Art. 58, (PI being the normal) $PI\times PF=BC^2=$ a constant quantity, $\therefore PI \propto \frac{1}{PF}$. By Art. 62. $CD\times PF=AC\times CB=$ a constant quantity, $\therefore CD \propto \frac{1}{PF}$; hence $PI \propto CD$. Again, since $CD \propto \frac{1}{PF}$, $\frac{CD^2}{PF}\left(\text{i. e. } CD^2\times\frac{1}{PF}\right)$ $\propto CD^3 \propto PI^3$; the radius of curvature therefore varies as PI^3, consequently the curvature itself varies as $\frac{1}{PI^3}$, or *inversely as the cube*

XIV.

On the Curvature of the HYPERBOLA.

The process for finding the chords and diameter of curvature in the Hyperbola is precisely the same as that for the Ellipse. Referring the reader to the annexed Figure, we shall merely repeat the principal steps of the foregoing demonstration.



(131.) *By Art. 111., $Pv \times vG : Qv^2 :: PC^2 : CD^2$, and at the point of contact, $QR \times 2PC : QP^2 :: PC^2 : CD^2$, $\therefore \frac{QP^2}{QR}$ or $PL =$

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of the normal. As $PI \propto CD$, the curvature varies as $\frac{1}{CD^3} \propto \frac{1}{DK^3}$, or *inversely as the cube of the diameter conjugate to that at the point of contact.*

* The construction of the above figure is word for word the same as in the Ellipse. To avoid a confusion of lines, the circle of curvature is drawn entirely within the Hyperbola; whereas, such part of the hyperbola as is of greater curvature than that at the point P, ought to have fallen within the circle of curvature, as in Fig. page 87.

$\frac{2CD^2}{PC}$ = chord of curvature passing through the center.

(132.) By similar triangles PCF, PLO, PO : PL$\left(\frac{2CD^2}{PC}\right)$ :: PC : PF, $\therefore$ PO $=\frac{2CD^2}{PF}$; and Pr the radius of curvature $=\frac{CD^2}{PF}$.

(133.) By similar triangles PEF, PVO, PV : PO$\left(\frac{2CD^2}{PF}\right)$ :: PF : PE(AC), $\therefore$ PV $=\frac{2CD^2}{AC}$ = chord of curvature passing through the focus.

(134.) At the vertex A, the diameter of curvature $\frac{2CD^2}{PF}=\frac{2BC^2}{AC}$ = latus-rectum. Here the analogy between the Ellipse and the Hyperbola ends; for with respect to the variation of curvature, since the normal PI keeps continually increasing from the point A,* the curvature will continually decrease as the point P recedes from A.

(135.) In the equilateral hyperbola (See Fig. in page 74) the latus-rectum is equal to the major axis; the curvature therefore at the vertex A is the same with the curvature of the circle described upon the major axis. In this case PI = PC (Art. 115); $\therefore$ $PI^3 \propto PC^3$, and in the recess of P from the point A, the curvature varies in the same ratio, viz. $\left(\frac{1}{PI^3} \text{ or } \frac{1}{PC^3}\right)$ with respect to the two sides of the isosceles triangle CPI, one of which (PC) revolves round the fixed point C, and the other (PI) round the moveable point I, at right angles to the curve. Here then is an instance of great symmetry in the curvature of the equilateral hyperbola.

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* That the radius of curvature varies as the cube of the normal, is proved in the same manner as in Note †, page 88.

CHAPTER VI.

ON THE ANALOGOUS PROPERTIES OF THE THREE CONIC SECTIONS.

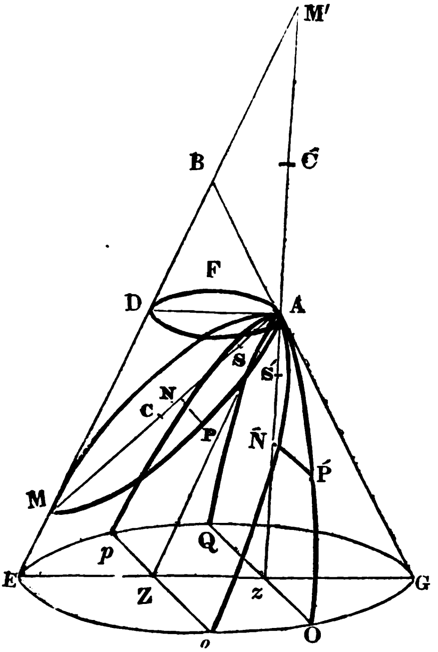
HITHERTO we have noticed no other analogies than those which take place between the Ellipse and Hyperbola; but as the three Conic Sections are derived from the same solid merely by changing the position of the plane which intersects its surface, it may naturally be expected that they will possess many properties common to them all. Previous to the investigation of these analogous properties, it may be worth while to consider the changes which take place in the nature of the section, during the revolution of the plane of intersection from a position parallel to the base of the cone, till it becomes a tangent to one of its sides.

XV.

On the changes which take place in the nature of the curve described upon the surface of a cone, during the revolution of the plane of intersection.

(136.) Let the triangle BEZG represent the section of a cone perpendicular to its base, and passing through the vertex; then if the cone be cut by a plane perpendicular to BEZG, and parallel to the base, the section AFD will be a circle. Draw the diameter AD of the circle AFD, and draw AZ parallel to the side BE of the cone. Conceive a plane (at right angles to the plane BEZG) to pass through AD, and afterwards to revolve through the angle DAG till it becomes a tangent to the side BG of the cone. From what was shown in Chapter I. it is evident that whilst this plane revolves

through the angle DAZ, its intersection APM with the surface of the cone will be an Ellipse, whose major axis is AM; when it comes



into the position AZ, it will be a Parabola, whose axis is AZ; and that whilst it revolves through the angle ZAG, it will be an Hyperbola, whose major axis is AM′, M′ being the intersection of zA and EB produced.

It may further be observed, that in the revolution of the plane through the angle DAZ, so long as it cuts the side BE between D, and E, a whole ellipse will be formed upon the surface of the cone. When it comes into such a position as to cut the base, a part only of an ellipse will be formed; and when it arrives at the position AZ, the point M moves off to an infinite distance, so that the Parabola thus formed may be considered as a part of an Ellipse, whose axis major is infinite. And as at the instant the plane leaves the position AZ in direction Zz, the curve of intersection becomes an Hyperbola,

the *Parabola* may also be regarded as an *Hyperbola*, whose major axis is infinite. These three curves therefore approach to identity at the same time that the plane approaches to parallelism with the side BE of the cone.

(137.) The same conclusion may be drawn from the algebraic construction of these curves. Let the angle MAZ be equal to the angle ZAz, then the major axis (AM′) of the Hyperbola will be equal to the major axis (AM) of the Ellipse.* In each case, find the center C or C′, and let the abscissas AN or AN′$=x$, the ordinate PN or P′N′$=y$, semi-axis major (AC or AC′)$=a$, semi-axis minor $=b$, AS or AS′ (S or S′ being focus)$=c$. Then in the Ellipse NM=AM−AN$=2a-x$, and MS=AM−AS$=2a-c$; in the Hyperbola, N′M′=AM′+AN′$=2a+x$, and M′S′=AM′+AS′$=2a+c$. Now by Arts. 46, 81. (see Figs. in pp. 32, 56.) we have, AN×NM or AN′×N′M′ : PN^2 or P′N′^{2} : : AC^2 : BC^2, or $x\times(2a\pm x) : y^2 :: a^2 : b^2$.

Hence $y^2=\frac{b^2}{a^2}\times(2ax\pm x^2)$ is the general equation between the abscissa and ordinate of the ellipse and hyperbola.

But in the Ellipse MS×SA=BC^2, and in the Hyperbola M′S′×S′A=BC^2, or $(2a\pm c)\times c=b^2$, hence by substitution $y^2=\frac{2ac\pm c^2}{a^2}\times(2ax\pm x^2)4cx\pm=\frac{2c^2x}{a}\pm\frac{2cx^2}{a}+\frac{c^2x^2}{a^2}$. Conceive now the angles MAZ, ZAz to be continually diminished, then the axis major both in the Ellipse and Hyperbola is continually increased, and just at the instant of their approach to coincidence with the line AZ, each of them becomes indefinitely great; in which case, (supposing x and c to be finite quantities) the three fractional terms of the last equa-

* Since AZ, is parallel to M′BE, the angles MAZ, ZAz are respectively equal to the angles of the triangle MAM′; which triangle is therefore isosceles.

tion become equal to nothing; $\therefore y^2=4cx$, or $PN^2=4AS\times AN$, which is the property of the Parabola. Hence it appears that a finite part of an Ellipse or Hyperbola whose latus-rectum is finite, but whose axis major is infinite, may be considered as a Parabola; and vice versa, that a finite part of a parabola may be considered as a part of an ellipse or hyperbola, whose axis major is infinite, and latus-rectum finite.

XVI.

On the mode of constructing the Three Conic Sections by means of a DIRECTRIX, *and the Properties derived therefrom.*

In Chap. I. we have already shown the method of constructing the Parabola by means of a directrix; we now proceed to show that the Ellipse and Hyperbola may also be constructed by lines revolving in a similar manner.

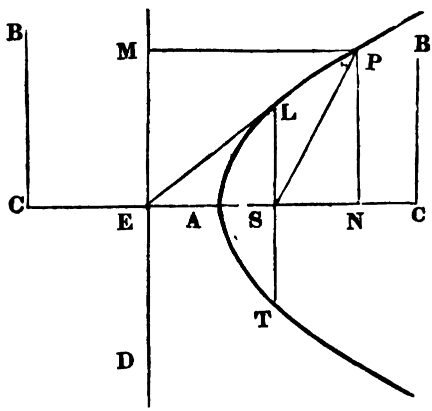
(138.) Let MED be a line given in position; and from the point E, draw CEC at right $\angle$ s to MED; in CEC take any point A, and set off $AS : AE :: m : 1$. Let the line SP begin to revolve from A round S, and PM move parallel to EC, in such manner that SP may be always to PM as AS to AE (i. e. in the given ratio of $m : 1$.); then the curve generated by the point of intersection P will be one of the Conic Sections.

Let fall PN at right $\angle$ s to AC, and let $AN=x$, $PN=y$, $AS=c$; then, since $AS\ (c) : AE :: m : 1$, we have $AE=\frac{c}{m}$; now $PM = NE=AE+AN=\frac{c}{m}+x$, and $SP : PM\left(\frac{c}{m}+x\right) :: m : 1$; $\therefore SP = c+mx$; also $SN=AN-AS=x-c$.

Hence we have,

$$\begin{aligned} SP^2 &= (c+mx)^2 = c^2+2cmx+m^2x^2, \\ SN^2 &= (x-c)^2 = c^2-2cx+x^2; \\ \hline \therefore SP^2-SN^2(=PN^2) &= y^2 = (m+1)2cx+(m^2-1)x^2. \end{aligned}$$

(139.) Let $m=1$, or $SP=PM$, then $m+1=2$, and $m^2-1=0$, $\therefore y^2=4cx$, or $PN^2=4AS\times AN$; hence ALP is a parabola, whose vertex is A, focus S, and axis AC.



(140.) Let m be less than 1, or SP less than PM. On the same side of A with PN, take $AC : SC :: 1 : m$, or $AC : AC-SC(=SA=c) :: 1 : 1-m$, then $c=(1-m).AC$; hence $(m+1)2cx=(m+1)(1-m)\times 2AC.x=(1-m^2).2AC.x$. From C draw BC at right angles to AC, and take $BC^2 : AC^2 :: 1-m^2 : 1$, then $1-m^2=\frac{BC^2}{AC^2}$, and $m^2-1=-\frac{BC^2}{AC^2}$. Substitute these values for $1-m^2$ and m^2-1, and we have $(m+1)2cx=\frac{BC^2}{AC^2}\times 2AC.x$, and $(m^2-1)x^2=-\frac{BC^2}{AC^2}.x^2$; now let $AC=a$, $BC=b$, then $y^2=((m+1)2cx+(m^2-1)x^2=)\frac{b^2}{a^2}\times(2ax-x^2)$. Hence by (Art. 137) ALP is an ellipse, whose semi-axis major $=AC$, semi-axis minor $=BC$, and focus S.*

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* To prove that S is the focus, we have $AC : SC :: 1 : m$, $\therefore AC^2 : SC^2 :: 1 : m^2$, and $AC^2 : AC^2-SC^2 :: 1 : 1-m^2$; but $AC^2 : BC^2 :: 1 : 1-m^2$, $\therefore BC^2=AC^2-SC^2$, and $SC^2=AC^2-BC^2$.

(141.) Let m be greater than 1, or SP greater than PM. Take C on the other side of A in such a manner that AC : SC::1 : m, or AC : SC $-$ AC($=$AS$=c$)::1 : $m-1$, then $c=(m-1)$.AC, and $(m+1)2c=(m+1)(m-1).2\text{AC}=(m^2-1).2\text{AC}$. From C draw BC at right $\angle$ s to AC, and take BC^2 : AC^2::m^2-1 : 1, then $m^2-1=\frac{\text{BC}^2}{\text{AC}^2}$. Let BC$=b$, AC$=a$, and substituting as before, we have $y^2=\frac{b^2}{a^2}(2ax+x^2)$; hence ALP is an hyperbola, whose semi-axis major is AC semi-axis minor BC, and focus S.

From this mode of describing the three Conic Sections we deduce the following properties.

Property 1.

If a tangent be drawn to the extremity of the latus-rectum of any conic section, it will cut the axis, or the axis produced, in the same point with the directrix.

Draw the latus-rectum LST; let LE be a tangent to the curve at L, and cut the axis in E; then

(142.) In the Parabola, SP$=$PM, $\therefore$ AS$=$AE; hence SE$=$ 2AS. But by Art. 18, the sub-tangent SE$=$twice the abscissa AS, $\therefore$ E is the extremity of the sub-tangent, and also a point in the directrix.

(143.) In the Ellipse, AS : AE::m : 1, m being less than 1.
By construction (140.) SC : AC::m : 1.
$\therefore$ (Euc. 12. 5.) SC : AC::SC$+$AS(AC) : AC$+$AE(EC).
Therefore EC is a third proportional to SC and AC; which is also true (55.) if E be the point where the tangent cuts the axis produced. Hence E is a point both in the directrix and tangent.

(144.) In the Hyperbola,
AS : AE::m : 1, m being greater than 1.

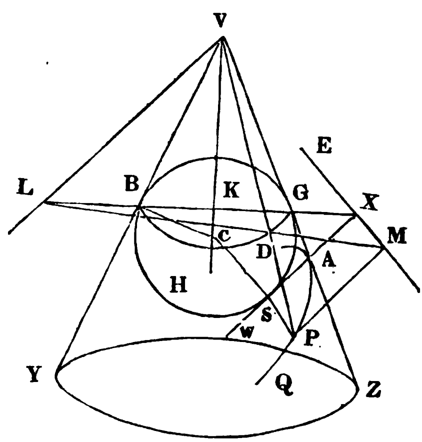
By construction (141.) SC : AC : : m : 1 ;
∴ (Euc. 19. 5.) SC : AC : : SC – AS(AC) : AC – AE(EC).

Therefore, EC is a third proportional to SC and AC; which is also true (85.) if E be the point where the tangent cuts the axis. Hence, E is a point both in the directrix and tangent.

(145.) This line LE, which is drawn touching the curve at the extremity of the latus-rectum, is called the focal tangent; from what has just now been proved, it follows therefore that if a line be drawn at right angles to the axis from the point where it is intersected by the focal tangent, that line will be the directrix.*

PROPERTY A.

(145.*a*.) In the cone VYZ, let APQ be any conic section, and BHG an inscribed sphere, touching the cone in the circle BDG and,



the plane of the conic section APQ in S. Then S is the focus of the conic section APQ. Also, if the plane of the circle BDG be

* The substance of Arts. 138 to 145, inclusive, may very readily be inferred, without the aid of Algebra, from Arts. 18.*a*., 57.*a*., and 87.*a*.

produced to intersect the plane of the conic section APQ in EF, then EF is the directrix of the conic section APQ.

Let VYZ be a plane passing through the axis VC of the cone, cutting the plane of the conic section APQ perpendicularly in AW, the axis of the conic section (1, 3 and 5,) and cutting the circle BDG in the line BG. Since VB and VG are tangents to the sphere from the same point V, they are equal* and the axis VC of the cone, which bisects the angle BVG, cuts BG at right angles. For the same reason, the axis VC cuts all other lines passing through K in the plane of the circle BDG at right angles, and this plane is, therefore, perpendicular to the axis VC, and consequently, to the plane VYZ, which passes through it. Since, therefore, the planes of the circle BDG, and the conic section APQ are both perpendicular to the plane VYZ, their common intersection EF is perpendicular to VYZ, and therefore to the lines BX, WX, which it meets in that plane.

Draw VL, in the plane VYZ, parallel to AW, intersecting GB, produced if necessary, in L. From any point P in the curve APQ, draw PM at right angles to EF. PM is parallel to WX, and consequently to VL. Join VP, intersecting the circumference of the circle BDG in D. Join LD, DM.

Since D is in the plane of the parallels PM, VL, the lines LD, DM are in that plane. But they are also in the plane of their circle BDG. Therefore they are in the common intersection of the two planes, and are in the same straight line. Now VD=VB, because both are tangents to the sphere from the same point V. For the same reason PS=PD.

And (sim. tri.) PD : PM : : VD : VL,
PS : PM : : VB : VL, a constant ratio.

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* For the plane of the lines VB VG cuts the sphere in a circle, to which VB and VG are tangents. Hence it follows from Euc. 36. 3. that VB=VG.

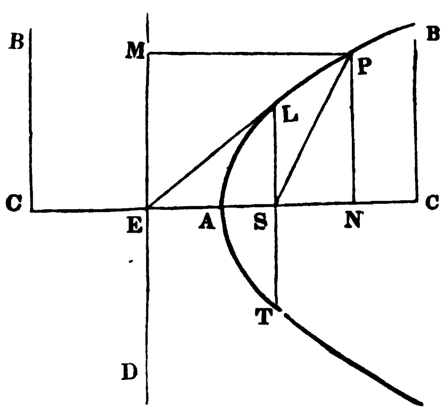
Hence, the distance SP, of any point of the curve P, from S, is in a constant ratio to the perpendicular PM, to the line EF; which is the property of the focus and directrix of the conic section APQ. Therefore S is the focus, and EF the directrix.

Property 2.

In any Conic Section, the distance SP=

$$\frac{\text{half latus-rectum}}{1-\frac{SC}{AC}\cdot\cos.\angle PSN}.$$

(146.) Let radius = 1, then SP : SN::1 : cos. $\angle$ PSN; therefore SN=SP$\times$cos. PSN. Now, in the Ellipse and Hyperbola, SP : PM::m : 1, and SC : AC::m : 1; $\therefore$ SP : PM(=NE=SE+SN)::SC : AC; hence SP$\times$AC=SE$\times$SC+SC$\times$SN=



$BC^2+SC\times SN=BC^2+SC\times SP\times\cos.PSN$; or $SP\times AC-SP\times SC\times\cos.PSN=BC^2$; therefore $SP=\frac{BC^2}{AC-SC\times\cos.PSN}$

$$=\frac{BC^2}{AC}\times\frac{1}{1-\frac{SC}{AC}\cdot\cos.\ PSN}=\frac{\text{half latus-rectum*}}{1-\frac{SC}{AC}\cdot\cos.\ PSN}.$$

(147.) In the Parabola, SC may be considered as equal to AC;† $\therefore SP=\frac{\text{half lat. rect.}}{1-\cos.\ PSN}$. The same expression might also be deduced immediately from the properties of the Parabola, for since $SE=2AS$, $SP(=PM=NE=SE+SN)=2AS+SN=2AS+SP\times\cos.PSN$, $\therefore SP-SP\times\cos.\ PSN=2AS$, and $SP=\frac{2AS}{1-\cos.\ PSN}=\frac{\text{half lat. rect.}}{1-\cos.PSN}$.

(148.) By means of this property we are enabled to find the variation of the distance SP in its angular motion round the focus S; and in this respect it forms an important theorem in Physical Astronomy. To put the expression just now deduced into the Algebraic form adopted by Mr. VINCE (at page 26 of his 'Physical Astronomy') in tracing the radius vector (SP) round the elliptic orbit of the moon, let $AC=1$, $BC=c$, $SC=w$, $\angle\ PSN=z$; then

---

* When P is at L, $\angle$ PSN = a right angle; $\therefore \cos.\ PSN=0$, and $SP=\frac{1}{2}$ latus-rectum. When P is between L and A, cos. PSN is negative; $\therefore SP=\frac{\frac{1}{2}\text{ latus-rectum}}{1+\frac{SC}{AC}\times\cos.PSN}$.

† For by Art. 137, the Parabola may be considered as an Ellipse, whose major axis is infinite; in this case C goes off to an infinite distance, and the difference (AS) between AC and SC vanishes with respect to the quantities themselves, which may therefore be assumed as in a ratio of equality.

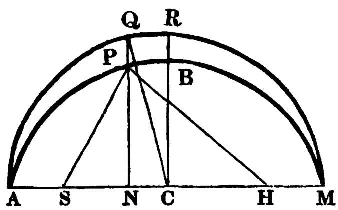
$$SP=\left(\frac{BC^2}{AC-SC\times\cos.\ PSN}=\right)\frac{c^2}{1-w\times\cos.\ z}$$

$$=c^2\times\frac{1}{1-w\times\cos.z}=(\text{by actual division})$$

$$c^2\times(1+w\times\cos.\ z+w^2\times(\cos.\ z)^2+w^3\times(\cos.\ z)^3+,\ \&c.)$$

For the trigonometrical transformation of this expression, and its practical application, we refer the reader to the work itself.

(149.) Before we leave this subject of the radius vector, it may not be improper to show its variation with respect to an angle described about the center of the Ellipse. Upon the major-axis AM describe the semi-circle AQM, produce NP to Q, and join QC.



Draw PH to the other focus, then $PN^2=SP^2-SN^2=HP^2-HN^2$; $\therefore HP^2-SP^2=HN^2-SN^2$.

Hence we have,

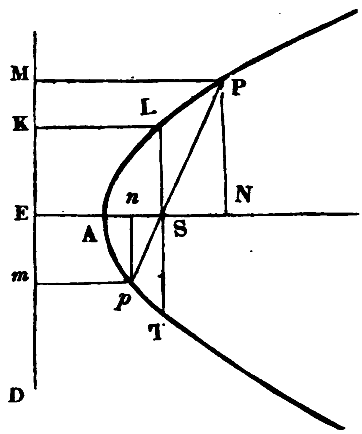
$$(HP+SP).(HP-SP)=(HN+SN).(HN-SN);$$
$$\therefore HP+SP : HN+SN :: HN-SN : HP-SP,$$
$$\text{i. e. } 2AC : HS \text{ or } 2SC :: 2CN : 2AC-2SP;$$
$$\text{or } AC : SC :: CN : AC-SP,$$
$$\therefore SC : AC-SP :: AC \text{ (or } QC) : CN$$
$$:: 1 : \cos.\ QCN;$$
$$\therefore SC\times\cos.\ QCN=AC-SP,$$

or $SP=AC-SC\times\cos.\ QCN$, which is an expression for the radius vector, with reference to the center.

Property 3.

(150.) If a conic section be cut through the focus (S) by a line (Pp) terminated at each extremity by the curve, then $4SP\times Sp=$ latus-rectum $\times Pp$.

From the extremity of the latus-rectum LST draw LK at right $\angle$ s to the directrix; and from p draw pm parallel, and pn perpen-



dicular, to the axis. From the nature of the construction we have,

$$\text{SP} : \text{PM (or NE)} :: \text{SL} : \text{LK (or SE)},$$
$$\text{or SP} : \text{SL} :: \text{NE} : \text{SE};$$
$$\therefore \text{SP}-\text{SL} : \text{SL} :: \text{NE}-\text{SE (SN)} : \text{SE},$$
$$\text{i. e. SP}-\text{SL} : \text{SN} :: \text{SL} : \text{SE};$$

for the same reason, $\text{ST}-\text{S}p\ (\text{SL}-\text{S}p) : \text{S}n :: \text{SL} : \text{SE}$;

$$\therefore \text{SP}-\text{SL} : \text{SL}-\text{S}p :: \text{SN} : \text{S}n :: (\text{by sim. } \Delta\text{s})\text{SP} : \text{S}p.$$

$$\text{Hence SP}\times\text{S}p-\text{SL}\times\text{S}p=\text{SL}\ \times\text{SP}-\text{SP}\times\text{S}p,$$
$$\text{or } 2\text{SP}\times\text{S}p=\text{SL}\ \times(\text{SP}+\text{S}p)=\text{SL}\times\text{P}p;$$
$$\therefore 4\text{SP}\times\text{S}p=2\text{SL}\times\text{P}p=\text{latus-rectum}\times\text{P}p.$$

(151.) Cor. 1. Hence $\frac{1}{\text{SP}}+\frac{1}{\text{S}p}=\frac{2}{\text{SL}}$. For since $\text{SL}\times(\text{SP}+\text{S}p)=2\text{SP}\times\text{S}p$; we have $\frac{\text{SP}+\text{S}p}{\text{SP}\times\text{S}p}\left(=\frac{1}{\text{S}p}+\frac{1}{\text{SP}}\right)=\frac{2}{\text{SL}}$.

Cor. 2. Since $SP-SL:SL-Sp::SP:Sp$, SP, SL and Sp are in harmonical proportion. Or, *half the latus-rectum is an harmonical mean between the segments, into which the focus of a conic section divides any line which passes through it.*

XVII.

On the analogous Properties of the Normal, Latus-rectum, Radius of Curvature, &c. &c. in all the Conic Sections.

If S be the focus, A the vertex, and P any point in the Parabola, then (Arts. 125, 126.) $4SP$=chord of curvature passing through the focus; $\frac{4SP^{\frac{3}{2}}}{\sqrt{(SA)}}=$ diameter, and $\frac{2SP^{\frac{3}{2}}}{\sqrt{(SA)}}=$ radius of curvature to the same point. In the Ellipse and Hyperbola, if C be center, S the focus, AC the semi-axis major, CD the semi-conjugate to the semi-diameter PC, and PF a perpendicular let fall from the point P to the conjugate diameter, then (Arts. 128, 129.) $\frac{2CD^2}{PC}=$ chord of curvature passing through the center from the point P; $\frac{2CD^2}{AC}=$chord through focus; $\frac{2CD^2}{PF}=$ diameter, and $\frac{CD^2}{PF}=$ radius of curvature to the same point.

Property 1.

In every conic section, the cube of the normal divided by the radius of curvature is equal to the square of half the latus-rectum.

(152.) In the Parabola, (see Fig. in page 22.) since normal $PO:SY::TP:TY$, and $TP=2TY$, $\therefore PO=2SY$; hence cube of normal $=8SY^3=$(by Art. 32.) $8SP^{\frac{3}{2}}\times SA^{\frac{3}{2}}$; radius of curvature $=\frac{2SP^{\frac{3}{2}}}{SA^{\frac{1}{2}}}$; $\therefore \frac{\text{cube of normal}}{\text{rad. of curvature}}=\frac{8SP^{\frac{3}{2}}\times SA^{\frac{3}{2}}\times SA^{\frac{1}{2}}}{2SP^{\frac{3}{2}}}=4SA^2=$ square of $2SA$=square of half the latus-rectum.

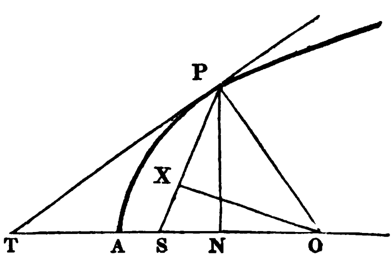
(153.) In the Ellipse and Hyperbola (see Figures and Properties in pp. 41, 59, 60.) $PO\times PF=BC^2$; $\therefore PO=\frac{BC^2}{PF}$, and cube of normal$=\frac{BC^6}{PF^3}$; radius of curvature$=\frac{CD^2}{PF}$; hence $\frac{\text{cube of normal}}{\text{rad. of curvature}}=\frac{BC^6\times PF}{CD^2\times PF^3}=\frac{BC^6}{CD^2\times PF^2}=$ (by Properties in pp. 43. 61) $\frac{BC^6}{AC^2\times BC^2}=\frac{BC^4}{AC^2}=$square of $\frac{BC^2}{AC}=$square of half the latus-rectum.

(154.) Cor. Since half the latus-rectum is a constant quantity, the radius of curvature varies as the cube of the normal; the curvature therefore varies inversely as the cube of the normal in all the Conic Sections; which accords with what has already been demonstrated in Sections XIII and XIV.

Property 2.

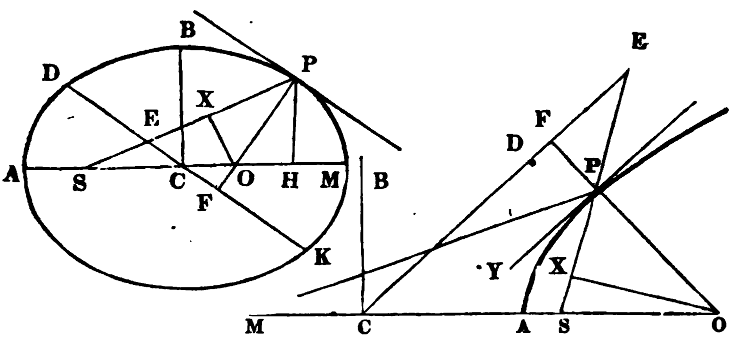
In any conic section, if a perpendicular (OX) be let fall upon the line SP from the point O, where the normal intersects the axis, then the part PX cut off by this perpendicular is equal to half the latus-rectum.

(155.) In the Parabola. Draw the ordinate PN; then, since (by Art. 30.) $SP=SO$, the angle $SOP=SPO$, and PO is common to the two right-angled triangles PXO PON; these two triangles are therefore equal and similar; hence $PX=NO=$(Art 21.) half the latus-rectum.



(156.) In the Ellipse and Hyperbola. (Fig. in p. 105.)

Draw the conjugate diameter DCK, then the right-angled triangles PEF, PXO are similar; $\therefore PE(AC):PF::PO:PX$; hence $AC\times PX=PO\times PF=BC^2$; $\therefore PX=\frac{BC^2}{AC}=$half the latus-rectum.

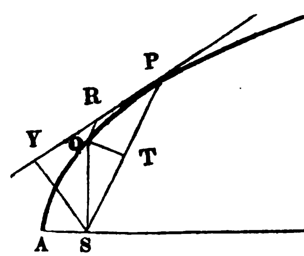


(157.) Cor. By means of this Property, if SP be given in length and position, and the latus-rectum and position of the tangent be also given, we can determine geometrically the position of the axis; for we have only to make PX equal to half the latus-rectum, and draw XO at right angles to SP, and PO at right angles to the tangent at P, then O (the intersection of XO, PO) is a point in the axis, which being joined to S, gives SO the position of the axis.

Property 3.

In any conic section, take the arc PQ, and from the point Q draw QT perpendicular and QR parallel to SP; then (assuming the relation of QT and QR just at the point of contact) the latus-rectum is equal to $\frac{QT^2}{QR}$.

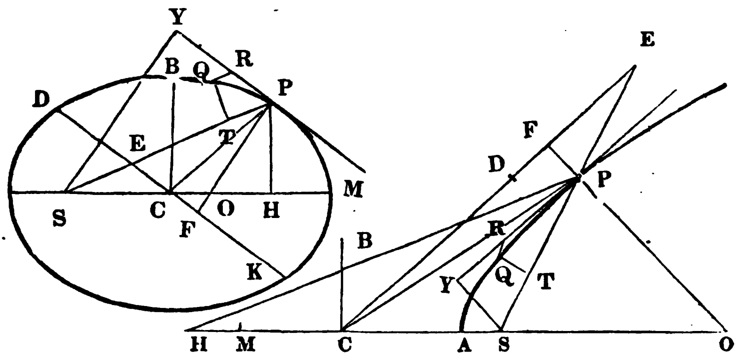
(158.) In the Parabola. Draw the perpendicular SY upon the tangent PY; then, since the arc QP coincides with the tangent at P, the triangle QPT continually approaches towards similarity with the triangle SPY as Q moves up towards P; and at the point of contact $QP : QT :: SP : SY$;



$\therefore QP^2 : QT^2 :: SP^2 : SY^2$, and (dividing the first two terms by

QR) $\frac{QP^2}{QR} : \frac{QT^2}{QR} :: SP^2 : SY^2 ::$ (by Art. 32.) $SP^2 : SP \times SA ::$ $SP : SA$. Now $\frac{QP^2}{QR}$ (=chord of curvature passing through the focus) $=4SP$; hence we have $4SP : \frac{QT^2}{QR} :: SP \therefore SA$, $\therefore \frac{QT^2}{QR} = \frac{4SP \times SA}{SP} = 4SA =$ latus-rectum.

(159.) In the Ellipse and Hyperbola. Draw the conjugate diameter DCK, and the perpendiculars PF and SY upon it and the tan-



gent; then the triangles QPT, PEF are similar, $\therefore QP^2 : QT^2 ::$ $PE^2(AC^2) : PF^2$, and $\frac{QP^2}{QR}\left(=PV=\frac{2CD^2}{AC}\right) : \frac{QT^2}{QR} :: AC^2 : PF^2$ $\therefore \frac{QT^2}{QR} = \frac{2CD^2 \times PF^2}{AC^3} = \frac{2AC^2 \times BC^2}{AC^3} = \frac{2BC^2}{AC} =$ the latus-rectum.

The demonstration of this property of the Conic Sections forms the substance of the first three Propositions of the third Section (B. 1.) of Sir Isaac Newton's Principia.

Property 4.

(160.) In every conic section, the chord of curvature passing through the focus is to the latus-rectum in the duplicate ratio of

SP : SY; and the diameter of curvature is to the same in the triplicate ratio of SP : SY.

For the chord of curvature passing through the focus $=\frac{QP^2}{QR}$; and, by Property 3, the latus-rectum $=\frac{QT^2}{QR}$; hence the

chord of curvature : latus-rectum :: $\frac{QP^2}{QR} : \frac{TQ^2}{QR}$ :: $QP^2 : QT^2$ :: $SP^2 : SY^2$;

but diameter : chord of curvature (see Fig. in page 87.) :: SP : SY;

$\therefore$ diameter of curvature : latus-rectum :: $SP^3 : SY^3$.

Property 5.

Let L=latus-rectum of any conic section; then, in the Parabola, $L\times SP=4SY^2$; in the Ellipse, $L\times SP$ is less than $4SY^2$; and in the Hyperbola, $L\times SP$ is greater than $4SY^2$.

(161.) In the Parabola. (32) $SA\times SP=SY^2$, $\therefore 4SA\times SP=4SY^2$, or $L\times SP=4SY^2$, for $L=4SA$.

(162.) In the Ellipse. By Art. 66. $SY^2=\frac{BC^2\times SP}{HP}$, $\therefore 4SY^2 =\frac{4BC^2\times SP}{HP}=\left(\text{for } \frac{2BC^2}{AC}=L, \text{ and } \therefore 4BC^2=2AC\times L\right) \frac{2AC\times L\times SP}{HP}$; hence $L\times SP : 4SY^2 :: HP : 2AC :: 2AC-SP$* : 2AC, and as $2AC-SP$ is less than 2AC, $L\times SP$ must be less than $4SY^2$.

(163.) In the Hyperbola, by a similar process we have $L\times SP : 4SY^2 :: HP : 2AC :: 2AC+SP$† : 2AC, and as $2AC+SP$ is greater than 2AC, $L\times SP$ must be greater than $4SY^2$.

* For $SP+HP=2AC$, $\therefore HP=2AC-SP$.

† For $HP-SP=2AC$, $\therefore HP=2AC+SP$.

(164.) Before we conclude this Section, it will be proper to show the method of expressing the relation between SP and SY, in the form of an algebraic equation. In the Parabola, therefore, let $SA=a$, $SP=x$, $SY=y$; then since $SY^2=SA\times SP$, we have $y^2=ax$, or $y=\sqrt{(ax)}$, for the equation to the curve, in terms of *the distance from the focus, and the perpendicular from the focus upon the tangent.* In the Ellipse and Hyperbola, let $AC=a$, $BC=b$, $SP=x$, $SY=y$; then $HP=2AC\pm SP=2a\pm x$, $\therefore$ since $SY^2=\frac{BC^2\times SP}{HP}$, we have $y^2=\frac{b^2x}{2a\pm x}=\frac{b^2x^2}{2ax\pm x^2}$, and $y=\frac{bx}{\sqrt{(2ax\pm x^2)}}$ where the negative or positive sign must be used according as the section is an Ellipse or an Hyperbola.*

(165.) To investigate the relation between CP and Cy (see Figures in pages 41, 60,) let $CP=x$, Cy or $PF=y$; then in the Ellipse, since $AC^2+BC^2=CD^2+PC^2$, we have $a^2+b^2=CD^2+x^2$, $\therefore CD^2=a^2+b^2-x^2$ or $CD=\sqrt{(a^2+b^2-x^2.)}$ Again since, $AC\times BC=CD\times PF$, we have $ab=CD\times y$, $\therefore CD=\frac{ab}{y}$; hence $\frac{ab}{y}=\sqrt{(a^2+b^2-x^2)}$, or $y=\frac{ab}{\sqrt{(a^2+b^2-x^2)}}$ is the equation to the curve in terms of the *distance from the center, and perpendicular from the center upon the tangent.*

In the Hyperbola, $PC^2 \backsim CD^2=AC^2\backsim CB^2$, or $x^2\backsim CD^2=a^2\backsim b^2$; $\therefore CD^2=x^2-a^2+b^2$, and $y=\frac{ab}{\sqrt{(x^2-a^2+b^2)}}$.†

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* These expressions are the equations of the several conic sections, considered as *spirals*, described by the revolution of the radius vector SP, about the focus.

† In these equations, the curves are considered as described by a radius vector CP, revolving about the center. This mode of consideration is, of course, inapplicable to the Parabola.

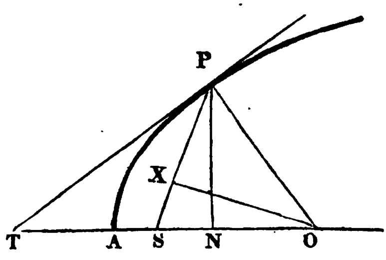
CHAPTER VII.

ON THE METHOD OF FINDING THE DIMENSIONS OF CONIC SECTIONS WHOSE LATERA-RECTA ARE GIVEN, AND OF DESCRIBING SUCH AS SHALL PASS THROUGH CERTAIN GIVEN POINTS.

XVIII.

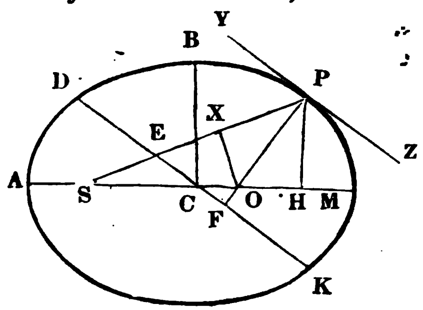
On the method of finding the dimensions of Conic Sections, whose latera-recta are given.

(166.) LET S be the focus of any conic section, P some point in the curve at a given distance from S; join SP, and let it meet the tangent PT in the given angle SPT; let the latus-rectum = L, and take PX = $\frac{1}{2}$L; from X draw XO at right angles to SP, and from P draw PO at right angles to PT, then by Art. 157, O will be a point in the axis; join SO, and it will give the position of the axis.

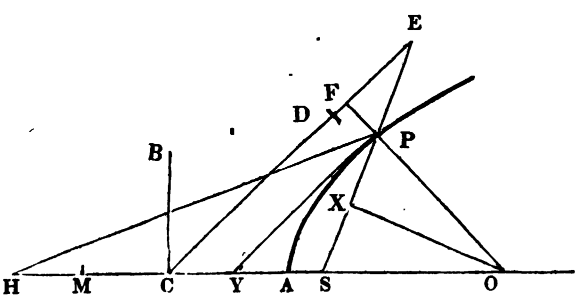


(167.) We are thus furnished with the means of determining geometrically the position of the axis of any conic section whose latus-rectum is given, and whose tangent at a given point meets a line drawn from the focus to that point, in a given angle. The position of the axis being found, its dimensions may be ascertained from the properties of each particular curve. In the Parabola, the latus-rectum is equal to four times the distance of the focus from the vertex; if therefore in OS produced, we take SA equal to $\frac{1}{4}$L, A will be the vertex of the Parabola. In the Ellipse and Hyperbola, it will be necessary to find the center, as also the major and minor axis; which is done in the following manner.

(168.) In the Ellipse, the lines drawn from the foci to any point in the curve make equal angles with the tangent at that point; if therefore the angle HPZ be made equal to the angle SPY, and SO be produced to meet PH in the point H, that point will be the other focus; and this determines the length (SP+PH) of the major axis. Now by Art. 45, the conjugate diameter DCK cuts off from SP a part equal to the semi-axis major; hence if PE be taken equal to $\frac{1}{2}$(SP+PH), and through E we draw DC parallel to the tangent at P, C will be the center of the ellipse. It only remains therefore to produce SH both ways, and make CA, CM each equal to PE, and we have AM the major axis of the curve. But (39) the latus-rectum is a third proportional to the major and minor-axis; the minor axis is therefore a mean proportional between the major axis and the latus-rectum; from C then draw BC at right angles to AM, make BC a mean proportional between AC and $\frac{1}{2}$L, and B will be the extremity of the minor axis; thus the dimensions of the ellipse are determined.

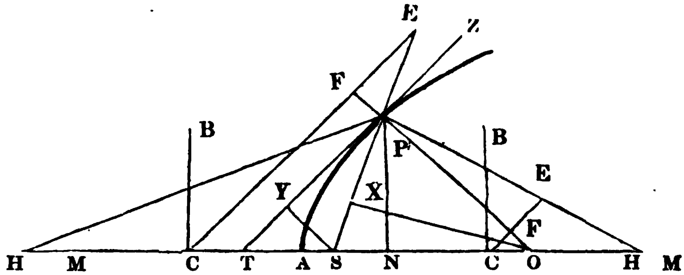


(169.) In the Hyperbola, the tangent bisects the angle SPH; in this case, therefore, the angle HPY must be made equal to the angle SPY on the opposite side of the tangent; then if OS is produced till it meets PH in the point H, that point will be the other focus. Produce SP to E, and take PE equal to $\frac{1}{2}$(HP−SP); through E



draw EC parallel to the tangent at P, and C will be the center. Take CA, CM, each equal to PE, then AM will be the major axis. The minor axis is determined precisely in the same manner as in the Ellipse.

(170.) We have thus shown the method of solving this Problem, when the nature of the curve is given. Suppose now that the latus-rectum, the distance SP, and the position of the tangent be given as before, and it is required to find not only the dimensions, but the nature of the conic section. In this case we have recourse to Arts. 161, 162, 163; from which, when the latus-rectum and the relation between SP and SY are given, we can determine the particular nature of the curve. For it is there proved, that if $L \times SP$ be equal to $4SY^2$, the curve is a Parabola; if $L \times SP$ be less than $4SY^2$, it is



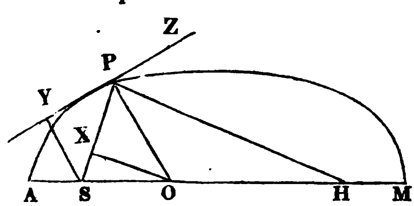
an Ellipse; and if $L \times SP$ be greater than $4SY^2$, it is an Hyperbola. In order to affect this general solution of the Problem, let the sine of the given $\angle$ $SPY = s$, radius $= 1$, then (by Trigonometry) $SP : SY :: 1 : s$; $\therefore SY = s \,.\, SP$, and $SY^2 = s^2 \,.\, SP^2$; consequently $4SY^2 = 4s^2 \,.\, SP^2$. Having therefore found the position of the axis, as in the former case; then, to know whether the conic section, whose dimensions are required, be a Parabola, Ellipse, or Hyperbola, we must compare $L \times SP$ with $4s^2 \,.\, SP^2$. If $L \times SP$ be equal to $4s^2 \,.\, SP^2$, i. e. if L be equal to $4s^2 \,.\, SP$, then the curve is a Parabola; take therefore $SA = \frac{1}{4}L$, and A is the vertex. If L be less than $4s^2 \,.\, SP$, the curve is an Ellipse; in which case, make the $\angle$ HPZ (on the same side of the tangent with SP) equal to SPY, and proceed as in Art. 168. If L be greater than $4s^2 \,.\, SP$, the curve is an Hyperbola; make therefore HPY (on the other side of the tangent) equal to SPY, and proceed as in Art. 169.

(171.) By Art. 160, the chord of curvature passing through the focus : the latus-rectum :: $SP^2 : SY^2 :: 1 : s^2$; $\therefore$ the latus-rectum $= s^2 \times$ chord of curvature; if therefore the chord of curvature and the relation of SP to SY be given, the latus-rectum will also be given. We are thus enabled to give the trigonometrical solution of the following

PROBLEM.

(172.) Given the chord of curvature passing from any point through the focus of a conic section, the distance of that point from the focus, and the position of the tangent; it is required to find the nature and dimensions of the conic section.

Let the chord of curvature to the point $P=40$, $SP=12$, the angle $SPY=30°$; then since the sine of $30°=$ half radius, $s=\frac{1}{2}$; $\therefore L=(s^2 \times \text{chord of curvature}=)\frac{1}{4}\times 40=10$; also $4s^2\times SP=4\times\frac{1}{4}SP=SP=12$; hence L is less than $4s^2\times SP$, and consequently the conic section is an Ellipse.



Since the $\angle$ $SPY=30°$, the $\angle$ $XPO=60$; $\therefore$ $\angle$ $XOP=30°$, and $PX=\frac{1}{2}PO$, or $PO=2PX=$(Art. 156.)$L=10$. Hence, in the triangle SPO, we have $SP=12$, $PO=10$, $\angle$ $SPO=60°$ from which we can determine the $\angle$ PSO; for $POS+PSO=120°$,* $\therefore \frac{1}{2}(POS+PSO)=60°$. Now $SP+PO$ (22) : $SP-PO$(2) :: tan. $\frac{1}{2}(POS+PSO)(60°)$: tan. $\frac{1}{2}(POS-PSO)=\frac{2\times \text{tan.}60°}{22}=\frac{\text{tan.}\ 60°}{11}$, $\therefore$ log. tan. $\frac{1}{2}$. $(POS-PSO)=$ log. tan. $60°-$ log. $11=$ log. tan. $8°\ 57'$; hence $\angle$ $PSO=(\frac{1}{2}(POS+PSO)-\frac{1}{2}(POS-PSO)=)60°-8°\ 57'=51°\ 3'$.

Since $\angle$ $HPZ=\angle$ SPY, the $\angle$ $OPH=\angle$ SPO, $\therefore$ $\angle$ $SPH=120°$; in the triangle SPH we have therefore $SP=12$, $\angle PSH=$

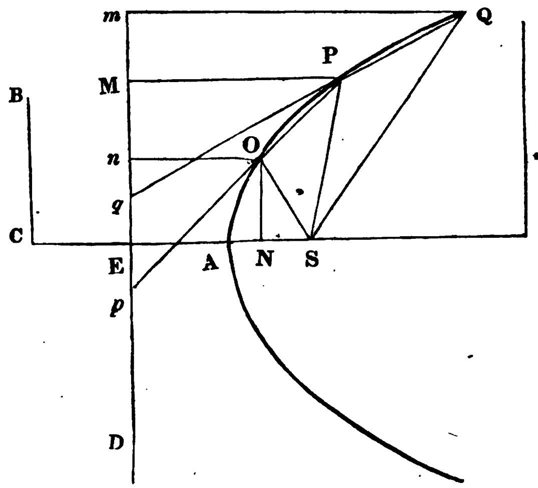
* See Day's Trigonometry, Art. 153.

$51^\circ\ 3'$, $\angle SPH=120^\circ$, and $\therefore \angle PHS=8^\circ\ 57'$; but as sin. PHS $(8^\circ\ 57')$* : sin. PSH$(51^\circ\ 3')$::SP(12) : $PH=\dfrac{12\times \sin.\ 51^\circ\ 3'}{\sin.\ 8^\circ\ 57'}$; hence log. PH=log. 12+log. sin. $51^\circ\ 3'$ − log. sin. $8^\circ\ 57'$=log. 59.987; $\therefore$ PH=59.987, and SP+PH=12+59.987=71.987= major axis of the ellipse, and the minor axis=(mean proportional between the major axis and latus-rectum=)$\sqrt{(10\times 71.987)}=26.83$; from which the Ellipse may be constructed as in Art. 168.

XIX.

On the method of describing Conic Sections which shall pass through three given points.

(173.) Let SO, SP, SQ, be three lines given in length and position; join PO, QP; produce PO to p, making Op : Pp::SO : SP; and produce it both ways to m and D. Draw SE, On, PM, Qm, at right angles to mED; then the conic section whose focus is S, directrix MED, and *determining ratio* SO::On, will pass through the points O, P, Q.



(174.) By sim. $\triangle$s Onp, PMp, Op : Pp::On : PM; but by the construction Op : Pp::SO : SP, $\therefore$ SO : SP::On : PM, and

* See Day's Trigonometry, Art. 150.

SO : On::SP : PM. Again, by sim. $\triangle$s PMq, Qmq, Pq : Qq:: PM : Qm; but Pq : Qq::SP : SQ.·.SP : SQ::PM : Qm, or SP : PM::SQ : Qm; hence SO : On::SP : PM::SQ : Qm, i. e. the lines SO, SP, SQ diverging from S are in a given ratio to the lines On, PM, Qm drawn at right angles to the line MED. By Art. 138, therefore the curve OPQ is a conic section whose focus is S and directrix MED; and it will be a parabola, ellipse, or hyperbola, according as the antecedent of that ratio is equal to, less or greater than, the consequent, or according as SO is equal to, less or greater than On.

(175.) In order to find the dimensions of the conic section; divide SE at the point A, so that SA : AE::SO : On, and A will be the vertex. If SO=On, then SA=AE, and the curve is a Parabola whose axis is EAS, vertex A, and latus-rectum 4SA. If SO be less than On, take AC : SC::On : SO, then (by Sect. XVI.) C will be the center and AC the semi-axis major of the Ellipse; the semi-axis minor (BC)=$\sqrt{(AC^2-SC^2)}$. If SO be greater than On, take C on the other side of A, so that AC : SC::On · SO, then C will be the center, and AC the semi-axis minor (BC)= $\sqrt{(SC^2-AC^2)}$.

This method of construction leads to the trigonometrical solution of the following

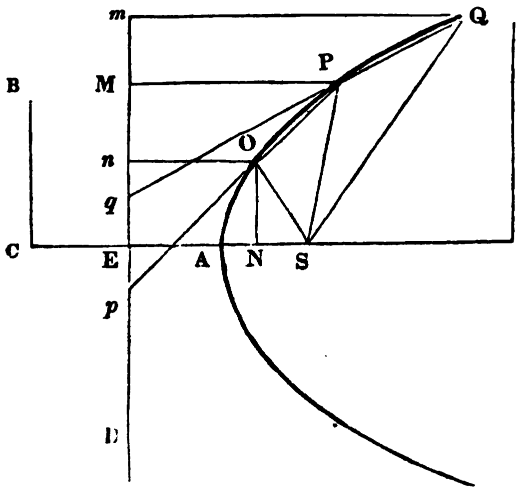
PROBLEM.

(176.) Three straight lines issuing from a point, being given in length and position; it is required to find the nature and dimensions of the conic section which shall pass through the extremities of those three straight lines, and have its focus in the point of their intersection.

Let SO=4, SP=7, SQ=10, $\angle$ OSP=60°, $\angle$PSQ=20°; then SOP+SPO=120°, and SP+SO (11)* : SP−SO (3) ::tan.

* See Day's Trigonometry, Art. 153.

$\frac{1}{2}$ (SOP+SPO) (60°) : tan. $\frac{1}{2}$ (SOP − SPO) $= \frac{3 \times \tan. 60°}{11}$, ∴ log. tan $\frac{1}{2}$. (SOP − SPO) = log. 3 + log. tan. 60° − log. 11 = log. tan. 25°



17′; hence ∠SOP = 60° + 25° 17′ = 85° 17′, and ∠SPO = 60° − 25° 17′ = 34° 43′.

Now, sin. SPO (34° 43′)* : sin. OSP (60°) : : SO (4) : PO $= \frac{4 \times \sin. 60°}{\sin. 34° \ 43'}$, ∴ log. PO = log. 4 + log. sin. 60° − log. sin. 34° 43′ = log. 6.0825, ∴ PO = 6.0825. By a similar process it appears that the angle SPQ = 125° 1′, ∠SQP = 34° 59′, and PQ = 4.1758. Thus all the sides and angles of the triangles SOP, SPQ are determined, and we now proceed to find the values of P*p* and Q*q*.

By the construction O*p* : P*p* : : SO : SP : : 4 : 7;
∴ O*p* : P*p* − O*p* (OP) : : 4 : 3.
Hence O*p* $= \frac{4}{3}$OP $= \frac{4}{3} \times 6.0825 = 8.11$;
∴ P*p* = O*p* + OP = 6.0825 + 8.11 = 14.1925.
Again, P*q* : Q*q* : : SP : SQ : : 7 : 10;
∴ P*q* : Q*q* − P*q* (PQ) : : 7 : 3.

* See Day's Trigonometry, Art. 150.

Hence $Pq=\frac{7}{3}\times PQ=\frac{7}{3}\times 4.1758=9.7435$;

$\therefore\ Qq=Pq+PQ=9.7435+4.1758=13.9193$.

Now the $\angle SPq=180^\circ-SPQ=180^\circ-125^\circ\ 1'=54^\circ\ 59'$; and the $\angle\ pPq=SPq-SPO=54^\circ\ 59'-34^\circ\ 43'=20^\circ\ 16'$; hence, in the triangle qPp, we have $Pq=9.7435$, $Pp=14.1925$, and the included angle $pPq=20^\circ\ 16'$; from which the angle qpP is found to be equal to $33^\circ\ 45'$. In the right-angled triangle Opn, we have therefore $Op=8.11$, and the $\angle Opn=33^\circ\ 45'$, which gives* $On=4.5056$.

Again in the triangle PpM, we have $Pp=14.1925$, and the angle $PpM=33^\circ\ 45'$, from which PM is found to be equal to 7.8849. Finally, by similar triangles, PqM, Qqm, we have Pq (9.7435) : Qq (13.9193)::PM(7.8849) : $Qm=\dfrac{Qq\times PM}{Pq}=11.264$.

On reviewing the steps of this operation, we have,

$$SO : On :: 4 : 4.5056 :: 1 : 1.1264,$$
$$SP : PM :: 7 : 7.8849 :: 1 : 1.1264,$$
$$SQ : Qm :: 10 : 11.264 :: 1 : 1.1264.†$$

The given ratio therefore of SO : On is 1 : 1.1262; and as, in this case, SO is less than On, the curve is an Ellipse.

Having thus ascertained the nature of the curve, it now only remains to find its dimensions. For this purpose we must first find the length of SE, which (if ON be let fall perpendicular to it) is equal to SN+NE, i. e. to SN+On, for On is equal to NE, being the opposite side of a parallelogram.

Now $\angle NOS=180^\circ-\angle SOP-\angle NOp(Opn)=180^\circ-85^\circ\ 17'-33^\circ\ 45'=60^\circ\ 58'$.

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* See Day's Trigonometry, Art. 134.

† It is not necessary to find all three of these ratios, since they are equal to one another. It would have been sufficient to have calculated the length of On, merely; for the ratio SO : On determines the nature of the curve.

In the triangle OSN we have therefore $OS=4$, and the angle $SON=60° 58'$, from which we get $SN=3.4973$; and consequently $SE(=SN+On)=3.4973+4.5056=8.0029$.

Having found the value of SE, we must divide it in the ratio of $SO : On$; i. e. in the given ratio of $1 : 1.1264$.

Thus $SA : AE :: OS : On :: 1 : 1.1264$;
$\therefore SA : SA+AE(SE) :: 1 : 2.1264$.

Hence $SA=\frac{SE}{2.1264}=\frac{8.0029}{2.1264}=3.7635$; therefore, make the angle $OSA=90° - 60° 58'=29° 2'$, and take $SA=3.7635$, then A will be the extremity of the major axis.

To find the major axis itself, take

$$AC : SC :: On : OS :: 1.1264 : 1,$$

or $$AC : AC - SC(AS) :: 1.1264 : .1264;$$

$$\therefore AC=\frac{1.1264}{.1264}\times AS,$$

$$=\frac{1.1264}{.1264}\times 3.7635=33.4951.$$

Hence $SC=AC-AS=33.4951-3.7635=29.7316$.
and $BC=\sqrt{(AC^2-SC^2)}=15.417$.

Finally the dimensions of the Ellipse are as follow:

$$\text{Major Axis}=2AC=66.9902.$$
$$\text{Minor Axis}=2BC=30.834.$$
$$\text{Latus-rectum}=\frac{2BC^2}{AC}=14.185.$$

By means of this Problem the dimensions of the orbit of a Planet or Comet may be found from three observations made as to its distance and angular position, at three different periods in the course of one revolution round the Sun.

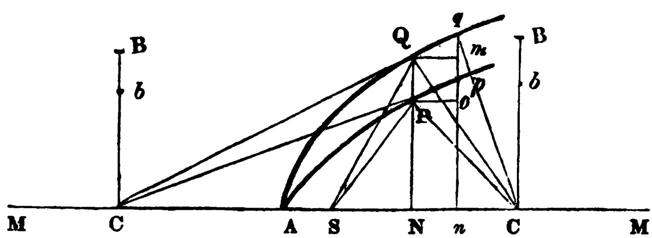
CHAPTER VIII.

ON THE QUADRATURE OF THE CONIC SECTIONS.

XX.

On the relation which obtains between the areas of Conic Sections of the same kind, having the same vertex and axis; and on the Quadrature of the Parabola, Ellipse, and Hyperbola.

(177.) Let AQ*q*, AP*p*, be any two curves having the same vertex A, and let them be referred to the same axis AM by ordinates



QN, PN, which are to each other in a given ratio; then the areas AQN, APN, generated by those ordinates, will be to each other in the same given ratio. For take the ordinate *qpn* indefinitely near to QPN, and draw Q*m*, P*o* parallel to the axis, then the fluxional or incremental areas QN*nq*, PN*np* will approach to equality with the parallelograms QN*nm*, PN*no*, as the ordinate *qpn* approaches to QPN; but (by Euc. 1. 6.) these parallelograms are to each other in the ratio of QN : PN; the nascent increments therefore of the areas AQN, APN are to each other in the ratio of QN : PN; and as these areas begin together from A, the areas themselves must also

be to each other in that ratio, i. e. area AQN : area APN :: QN : PN.

(178.) Suppose now the curves AQq, APp to be two Conic Sections of the same kind whose latera-recta are respectively L and l; for instance, let them be two Parabolas; then by the property of the parabola $L \times AN = QN^2$, and $l \times AN = PN^2$ hence AQN : APN (:: QN : PN) :: $\sqrt{(L \times AN)} : \sqrt{(l \times AN)} :: \sqrt{L} : \sqrt{l}$. If they be Ellipses or Hyperbolas which have the same major axis AM, and whose minor axes are respectively BC and bC, then

$$\begin{aligned} AN \times NM : \ PN^2 \ &:: AC^2 : bC^2, \\ \text{and } QN^2 : AN \times NM &:: BC^2 : AC^2; \\ \therefore QN^2 : \ PN^2 \ &:: BC^2 : bC^2 :: \frac{2BC^2}{AC} : \frac{2bC^2}{AC} \\ &:: L : l. \end{aligned}$$

Hence in this case also AQN : APN(:: QN : PN) :: $\sqrt{L} : \sqrt{l}$; i. e. in Conic Sections of the same kind, having the same vertex and axis, the areas AQN, APN are to each other in the given subduplicate ratio of their latera-recta.

(179.) Take any point S in the axis, and join SQ, SP; then we have

$$\begin{aligned} \text{area } AQN : \ &\text{area } APN :: QN : PN, \\ \text{and } \triangle\, SQN : \ &\triangle\, SPN \ :: QN : PN; \\ \hline \therefore AQN - SQN : \ &APN - SPN :: QN : PN, \\ \text{or area } AQS : \ &\text{area } APS :: QN : PN \\ &:: \sqrt{L} : \sqrt{l} \text{ in all the Conic} \end{aligned}$$

Sections.

(180.) But Ellipses and Hyperbolas having the same vertex and axis, will also have the same center.* Let C be that center, and in each case join QC, PC; then

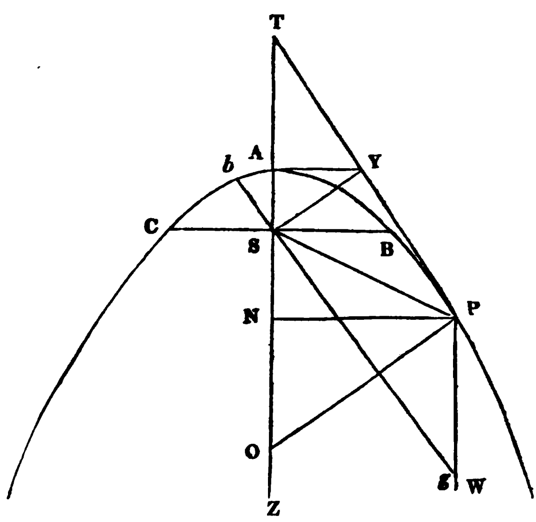
* Although they have the same center, it should be recollected that (since the minor axes are not equal) they cannot have the same focus.

In the Ellipse, AQN : APN :: QN : PN,
and $\triangle$QCN : $\triangle$PCN :: QN : PN;
$\therefore$ AQN+QCN : APN+PCN :: QN : PN,
or sector ACQ : sector ACP :: QN : PN,
:: $\sqrt{L} : \sqrt{l}$.

In the Hyperbola, AQN : APN :: QN : PN,
and $\triangle$ QCN : $\triangle$PCN :: QN : PN;
$\therefore$ QCN − AQN : PCN − APN :: QN : PN,
or sector ACQ : sector ACP :: QN : PN,
:: $\sqrt{L} : \sqrt{l}$.

On the Quadrature of the PARABOLA.

(181.) Let AP be a Parabola whose latus-rectum (BC)$=a$, abscissa (AN)$=x$, ordinate (PN)$=y$; then, by the Property of the Parabola, $y^2=ax$, $\therefore y=a^{\frac{1}{2}}x^{\frac{1}{2}}$, and $y\dot{x}=a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}$ now the fluent of $y\dot{x}=\frac{2}{3}a^{\frac{1}{2}}x^{\frac{3}{2}}=\frac{2}{3}x\times a^{\frac{1}{2}}x^{\frac{1}{2}}=\frac{2}{3}xy$; hence the area APN$=\frac{2}{3}AN\times$PN$=$ $\frac{2}{3}$ the circumscribing parallelogram.



(182.) Draw a tangent to the point P, and produce NA to meet it in T; then since $AN=\frac{1}{2}NT$, the $\triangle PNT=(\frac{1}{2}TN\times PN=)AN\times PN$; hence the area $ANP=\frac{2}{3}AN\times PN=\frac{2}{3}\triangle PNT$. Now suppose, in the Figure. at page 20, that a tangent be drawn to the point G, and that the line MS drawn parallel to the axis meets it in S, then the area $AZG=\frac{2}{3}\triangle TZG$; but $\triangle TZG : \triangle SMG :: ZG^2 : MG^2 :: 1 : 4$, $\therefore \triangle SMG=4$ times $\triangle TZG=6$ times area $AZG=3$ times area MAG; hence area $MAG=\frac{1}{3}\triangle SMG$.

(183.) But the area of a Parabola may ascertained in terms of the square of its latus-rectum. For let $AN : AS :: n : 1$, then $AN=n.AS$: but $PN^2=4AS\times AN=4n.AS^2$, $\therefore PN=2AS.\sqrt{n}$;

hence area $APN(\frac{2}{3}AN\times PN)=\frac{2}{3}\times n.AS\times 2AS\times\sqrt{n}=\frac{4n^{\frac{3}{2}}}{3}\times AS^2=$ (for $AS=\frac{1}{4}BC$, and $\therefore AS^2=\frac{1}{16}BC^2$) $\frac{n^{\frac{3}{2}}}{12}\times BC^2$; or if the whole Parabola is taken (as in Fig. p. 20), then the area $MAG=\frac{n^{\frac{3}{2}}}{6}\times$ square of latus-rectum.

(184.) Not only the area ANP contained between the abscissa and ordinate, but also the area ASP described by the revolution of the line SP round the focus S, may be ascertained in the same manner. For since $AN=n.AS$, $SN=(AN-AS=)(n-1).AS$; hence $\triangle SPN=(\frac{1}{2}SN\times PN=)\frac{n-1}{2}\cdot AS\times PN$. Now area APS=area $APN-\triangle SPN=\frac{2}{3}n.AS\times PN-\frac{n-1}{2}\times AS\times PN=\frac{n+3}{6}\cdot AS\times PN=$ (for $PN=2AS\sqrt{n}$) $\frac{(n+3).\sqrt{n}}{3}\times AS^2=\frac{(n+3).\sqrt{n}}{48}\times BC^2$.

(185.) Hence it appears, that if the latus-rectum be given, the parabolic areas ANP, ASP may be found without any other irrationality

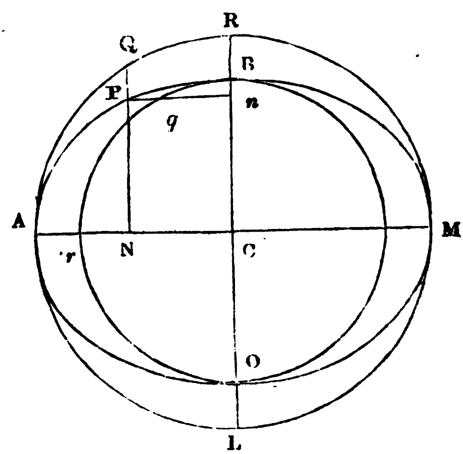
than that which arises from extracting the square root of numbers; for if $n=1, 4, 9, 16.$ &c. then

Area ANP$=\frac{1}{12}, \frac{2}{3}, \frac{9}{4}, \frac{16}{3}$, &c. of the square of the latus-rectum; and

Area ASP$=\frac{1}{12}, \frac{7}{24}, \frac{3}{4}, \frac{19}{12}$, &c. of the same; but if n be not a square number, then the expression for these areas will involve an irrational quantity.

On the Quadrature of the Ellipse.

(186.) Let ABMO be an Ellipse, and upon the major axis AM describe the circle ARML; draw any ordinate QPN, then by Property 9, of the Ellipse, QN : PN in the given ratio of RC or AC : BC. But from what was proved in Sect. 20, area AQN : area APN : : QN : PN : : AC : BC; and for the same reason, the semicircle ARM will be to the semi-ellipse ABM in the same ratio; hence the whole Ellipse ABM : circle ARML described upon its major axis : : BC : AC : : minor axis : major axis.



(187.) As the area of the Ellipse bears this given ratio to the area of its circumscribing circle, the quadrature of the Ellipse must therefore depend upon the quadrature of the circle. Let

$p=3.1416$ (=* area of a circle whose radius is 1), then the area of the circle whose radius is $AC=p\times AC^2$; hence the area of the Ellipse : $p\times AC^2::BC:AC$, $\therefore$ area of the Ellipse $=p\times AC\times BC$, i. e. the area of an Ellipse is found by multiplying the rectangle under its semi-axes by the same decimal number (p) as the square of the radius is multiplied by, to find the area of a circle. From this it also appears, that the area of an Ellipse is equal to the area of a circle whose radius is a mean proportional† between its semi-axes; for the area of that circle is equal to ($p\times(\text{rad.})^2=p\times$ the square of $\sqrt{(AC\times BC)}=$)$p\times AC\times BC$.

(188.) The area of the parallelogram circumscribing the Ellipse is equal to $4AC\times BC$, $\therefore$ area of Ellipse : area of that parallelogram$::p\times AC\times BC:4AC\times BC::p$, or $3.1416:4::.7854:1$; i. e. the area of an Ellipse has the same ratio to the area of its circumscribing parallelogram as the area of a circle has to its circumscribing square.

On the Quadrature of the Hyperbola.

(189.) Let APp be an Hyperbola whose semi-axis major $AC=a$, semi-axis minor $bC=b$; and let $CN=x$, $PN=y$; then by Cor. 1. Prop. 6. of Hyperbola, $CN^2-CA^2:PN^2::AC^2:BC^2$, or $x^2-a^2:y^2::a^2:b^2$, $\therefore y=\frac{b}{a}\sqrt{(x^2-a^2)}$; hence $y\dot{x}=\frac{b}{a}\cdot\dot{x}\sqrt{(x^2-a^2,)}$ whose fluent found by a series and properly corrected would give the value of the area APN; but this area may be ascertained by means of logarithms, when we have found the value of the hyperbolic sector ACP. (See Fig. in p. 118.)

(190.) Now the area of this sector is thus found. The area of $\triangle CPN=\frac{1}{2}CN\times PN=\frac{xy}{2}$, $\therefore$ the fluxion of the $\triangle CPN=\frac{x\dot{y}+y\dot{x}}{2}$;

* See Day's Mensuration, &c. Art. 30.

† Let $x=$ mean proportional between AC and BC, then $AC:x::x:BC$, $\therefore x^2=AC\times BC$, or $x=\sqrt{(AC\times BC)}$.

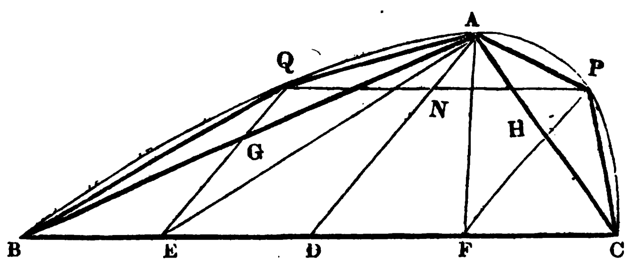
but sector ACP $=\triangle$ PCN $-$ area APN, $\therefore$ fluxion of sector ACP $=$ fluxion of $\triangle$ PCN $-$ fluxion of area APN $=\frac{x\dot{y}+y\dot{x}}{2}-y\dot{x}=\frac{x\dot{y}-y\dot{x}}{2}$; we must therefore find the values of $\frac{x\dot{y}}{2}$ and $\frac{y\dot{x}}{2}$. Since $y=\frac{b}{a}\sqrt{(x^2-a^2)}$, $\dot{y}=\frac{bx\dot{x}}{a\sqrt{(x^2-a^2)}}$, $\therefore \frac{x\dot{y}}{2}=\frac{bx^2\dot{x}}{2a\sqrt{(x^2-a^2)}}$; by Art. 189. $\frac{y\dot{x}}{2}=\frac{b\dot{x}\sqrt{(x^2-a^2)}}{2a}$; hence $\frac{x\dot{y}-y\dot{x}}{2}$ or fluxion of sector ACP $=\frac{bx^2\dot{x}}{2a\sqrt{(x^2-a^2)}}-\frac{b\dot{x}\sqrt{(x^2-a^2)}}{2a}=\frac{ab\dot{x}}{2\sqrt{(x^2-a^2)}}$, $\therefore$ the fluent or sector ACP $=\frac{ab}{2}\times$ hyp. log. $(x+\sqrt{(x^2-a^2)})+$Cor.; when $x=a$, ACP $=0$, $\therefore$ sector ACP $=\frac{ab}{2}\times$ hyp. log. $\frac{x+\sqrt{(x^2-a^2)}}{a}$.

(191.) The triangle CPN $=\frac{xy}{2}=\frac{bx\sqrt{(x^2-a^2)}}{2a}$, $\therefore$ area APN ($=\triangle$ CPN $-$ sector ACP) $=\frac{bx\sqrt{(x^2-a^2)}}{2a}-\frac{ab}{2}\times$ hyp. log. $\frac{x+\sqrt{(x^2-a^2)}}{a}$. Suppose AQ$q$ to be an equilateral hyperbola, in which $a=b=1$, then the area AQN $=\frac{1}{2}x\sqrt{(x^2-1)}-\frac{1}{2}$ hyp. log. $(x+\sqrt{(x^2-1)})$. A portion of this hyperbola, whose abscissa is equal to its semi-axis major (in which case $x=2$) will be numerically expressed by the quantity $\sqrt{3}-\frac{1}{2}$ hyp. log. $(2+\sqrt{3})=1.7320-.6584=1.0736$; thus in Figure page 74, if the abscissa AN be taken equal to AC, then the area (APN) corresponding to this abscissa : square ACBa : : 1.0736 : 1, and area APN : quadrant ACB : : 1.0736 : .7854 : : 1.3669 : 1.

XXI.

On the Quadrature of the PARABOLA, *according to the method of the Ancients.*

(192.) Let BQAPC be any portion of a Parabola cut off by the straight line BC; bisect BC in the point D, and draw DA parallel to the axis; then AD will be the diameter to the point A, and (by converse of Art. 23.) BC will be an ordinate to that diameter. Moreover, since a tangent to the point A is parallel to BC, A will be the highest point or vertex of the figure BQAPC; if therefore BA, AC, be joined, then this figure and the triangle ABC will have the same base and vertex.



(193.) Bisect BD in E, and draw EQ parallel to DA; through Q draw QNP parallel to BC, and from P draw PF parallel to AD; then QNP will be an ordinate to the diameter AD in the point N, and QE, PF will be diameters to the points Q, P respectively; and since QEDN is a parallelogram, QN will be equal to ED, i. e. to $\frac{1}{2}$BD; hence $QN^2 : BD^2 :: 1 : 4$; but by the property of the Parabola, $AN : AD :: QN^2 : BD^2$, $\therefore AN : AD :: 1 : 4$, or $AN = \frac{1}{4}AD$; hence ND or $QE = \frac{3}{4}AD$. Again, since EG is parallel to DA, and $BE = \frac{1}{2}BD$, EG must be equal to $\frac{1}{2}AD$, $\therefore QG = \frac{1}{4}AD$, and $EG : GQ :: 2 : 1$.

(194.) Join AE, AQ, QB; then since BD is bisected in E, the triangle ABE is equal to half the triangle ABD (by Euc. 1. 6.); and since GQ is equal to $\frac{1}{2}$GE, the triangles AQG, BQG are respectively half of the triangles AGE, BGE; hence the triangle AQB is half of the triangle ABE, and consequently $\frac{1}{4}$th of the triangle ABD. In the same manner (if AP, PC, AF, be joined,) it may

be proved that the triangle APC is $\frac{1}{4}$th the triangle ADC; hence the two triangles AQB, APC, taken together, are equal to one fourth of the triangle ABC.

(195.) Now suppose BE, ED were bisected, and from the points of bisection lines were drawn parallel to DA (which will evidently bisect BG, GA,) then the sum of the triangles formed within the parabolic spaces* BQ, QA (by drawing lines from the points where those parallel lines cut the curve to the extremities of the chords BQ, QA) will be equal to $\frac{1}{4}$th of the triangle AQB†; and the sum of the triangles formed in a similar manner within the parabolic spaces AP, PC, will be equal to $\frac{1}{4}$th of the triangle APC; $\therefore$ the sum of the triangles formed within the four parabolic spaces BQ, QA, AP, PC is equal to $\frac{1}{4}$th of $\triangle AQB + \triangle APC$, i. e. to $\frac{1}{16}$th of the triangle ABC. By bisecting the halves of BE, ED, &c. and drawing lines as before, parallel to DA, and joining the points of their intersection with the curve to the extremities of the chords, a series of eight triangles would be formed in the remaining parabolic spaces, the sum of which would be equal to $\frac{1}{4}$th of the sum of the triangles formed within the parabolic spaces BQ, QA, AP, PC, i. e. to $\frac{1}{64}$th of the triangle ABC. We might thus go on bisecting the successive parts of the base BC, and forming triangles in a similar manner, till the whole parabolic figure BAC was exhausted, in which case it is evident that the area of that figure would be equal to the sum of the areas of all the triangles thus formed within it.

(196.) Let the triangle ABC$=a$, then to find the sum of the areas of all these triangles, we have merely to sum the series

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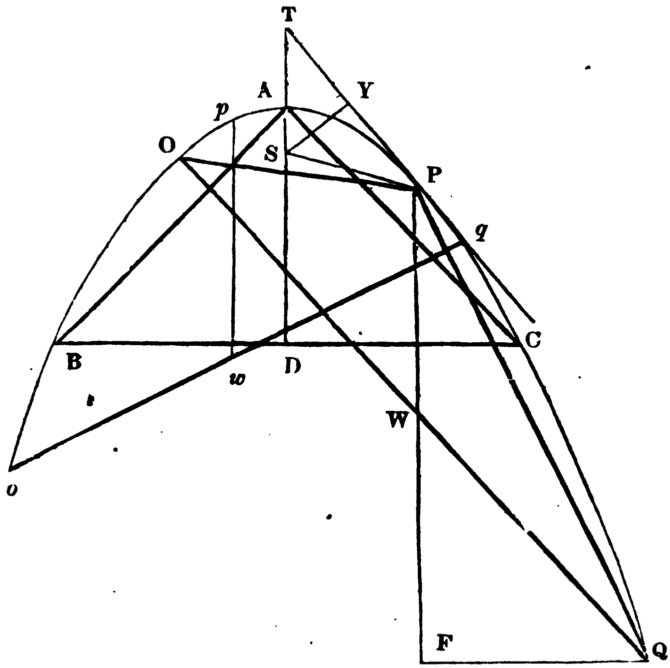
* By parabolic spaces, we mean such portions of the Parabola as are contained between the arcs BQ, QA, AP, PC, and the straight lines BQ, QA, AP, PC respectively.

† For the same reason that the sum of the triangles AQB, APC is equal to $\frac{1}{4}$th the triangle ABC, this conclusion being evidently true for the triangles thus inscribed in any portion of a Parabola.

$a+\frac{a}{4}+\frac{a}{16}+\frac{a}{64}+$&c. continued ad infinitum, which is a geometric series, whose first term is a, and common ratio $\frac{1}{4}$. Now the sum of this series* $=\left(\frac{a}{1-r}=\right)\frac{a}{1-\frac{1}{4}}=\frac{4a}{3}$; $\therefore$ the area of the parabola BAC is equal to $\frac{4}{3}\times$ area of the $\triangle$ BAC. If a tangent was drawn to the point A, and from B, C, lines were drawn parallel to DA, then the triangle ABC would be the half of the parallelogram thus formed; the parabolic area BAC is therefore $\frac{2}{3}$ds of the circumscribing parallelogram; which accords with what has already been proved respecting the quadrature of the Parabola in Section XX; for it is evident the foregoing demonstration is true for the axis, since AD is any diameter.

(197.) From the given ratio which subsists between the parabolic area and its inscribed triangle, we may prove, that such portions of a Parabola as are cut off by ordinates to equal diameters, are equal to one another. Let oAQ (Fig. in p. 128) be any Parabola, and draw the diameters PW, pw to the points P, p; take PW$=pw$, and through W, w, draw the ordinates OWQ, owq; draw the axis AD; take AD equal to PW or pw, and through D draw the ordinate BC; and in the parabolic spaces BAC, OPQ, inscribe the triangles BAC, OPQ. Draw the tangent to the point P, and produce the axis to meet it in the point T; let S be the focus, and join SP; from S let fall SY perpendicular upon the tangent, and draw QF perpendicular upon PW produced. Now $4SA\times AD=CD^2$, and $4SP\times PW=WQ^2$; therefore $WQ^2 : CD^2 :: 4SP\times PW : 4SA\times AD ::$ (since PW$=$AD) SP : SA. Again, since the ordinate WQ is parallel to the tangent TP, and the diameter PW is parallel to the axis AD, the triangles WQF, STY are similar, $\therefore WQ^2 : QF^2 :: ST^2$ (or SP^2) : $SY^2 ::$ (Euc. Def. 11. 5.) SP : SA; hence $WQ^2 : CD^2 :: WQ^2 : QF^2$, $\therefore$ CD$=$QF. But the $\triangle$ PWQ$=\frac{1}{2}$PW$\times$QF and the $\triangle$ ADC$=\frac{1}{2}$AD$\times$CD; since, therefore PW, QF, are respectively equal to AD, CD, the $\triangle$ PWQ must be equal to the $\triangle$ ADC.

* See Day's Algebra Art. 442.



Now these △s are the halves of the triangles OPQ and ABC; hence the △ OPQ is equal to the triangle ABC, and consequently the parabolic area OPQ to the parabolic area BAC.* In the same manner it might be proved that the parabolic area *opq* is equal to the area BAC; ∴ the area *opq* is equal to the area OPQ.

These observations upon the quadrature of the Parabola according to the method of the Ancients, contain the substance of the last seven propositions (viz. from 18 to 24 inclusive) of Archimedes *De Quadratura Parabolæ*, and of the fourth Proposition of his book *De Conoidibus et Sphæroidibus.*

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* For by Art. 195, the parabolic areas OPQ, BAC are $\frac{4}{3}$ds of the triangles OPQ, BAC respectively.

THE END.